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PARAMETER ESTIMATION SUPPLEMENT

to the

Mission Analysis Evaluation and Space
Trajectory Operations Program
MAESTRO

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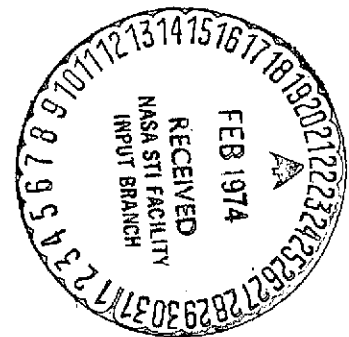
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Contract No. NAS 5-11900
August 1973

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INTRODUCTION

This Parameter Estimation Supplement describes the PEST computer program and gives instructions for its use in determination of lunar gravitation field coefficients.

PEST was developed under Contract NAS 5-11900 for use in the RAE-B lunar orbiting mission as a means of lunar field recovery. The observations processed by PEST are short-arc osculating orbital elements. These observations are the end product of an orbit determination process obtained with another program. PEST's end product is a set of harmonic coefficients, C_{nm} and S_{nm} , to be used in long-term prediction of the lunar orbit. PEST employs some novel techniques in its estimation process, notably a "square-root" batch estimator and linear variational equations in the orbital elements (both osculating and mean) for measurement sensitivities.

Section 1 describes the program's capabilities, giving operating instructions and input/output examples. This section supplements the User's Manual for the MAESTRO program (Ref. 1). PEST utilizes MAESTRO routines for its trajectory propagation. Section 2 describes PEST's program structure and contains descriptions of those of its subroutines which are not common to MAESTRO. The MAESTRO subroutines are described in the Programmer's Manual for MAESTRO (Ref. 2). Section 3 contains some of the theoretical background information for the estimation process. Appendix A contains a derivation of linear variational equations for the Method 7 elements.

I. PEST PROGRAM DESCRIPTION (User's Guide)

1.1 Highlights and Capabilities

The parameter estimation (PEST) program is a weighted least squares batch processor which estimates

- a. coefficients of the lunar gravitational field
- b. initial orbital elements
- c. solar pressure coefficients.

The measurements used in the estimation are short-arc determinations of the orbital elements relative to the Moon. PEST can process all or any subset of the residuals in the elements. The elements can be osculating or averaged values of either classical form or Gaussian.

Methods 4 or 5: $p, e, M, \omega, i, \Omega$

Method 7: $p, e \sin \omega, e \cos \omega, \omega + f, i, \Omega$

Method 8: $p, e \sin \omega, e \cos \omega, \omega + M, i, \Omega$

The measurement data can be

- a. read in from magnetic tape
- b. read in on cards
- c. simulated by the program itself

Measurement sensitivities can be generated either by the secant method or by integration of linear variational equations. The sensitivities may be "frozen" at some iteration of the estimation process, thus saving the computation time needed to recompute sensitivities. The estimated state increment may be included with a gain factor other than unity to combat nonlinearity problems. Selected parameters may be held fixed during the estimation while they are being "considered" in the calculation of the state increment.

1.2 Operating Instructions

Figure 1.1 illustrates the order in which PEST accepts and expects input (card) data. The READ statements occur in only two subroutines, INPUTF and FILTER. INPUTF reads data into INPUT common, reading three (address, value) pairs per card with a (1X,3(I5,D15.8)) format. A blank or zero in the first address field causes a return from INPUTF.

The first or only card read by FILTER is to identify the parameters to be estimated and to flag the primary options of this capability. The next card is a measurement selector. FILTER next expects measurement data. If an end of file is encountered at this point, the (only) case will be executed. If another case is to follow the current one, an F10.2 zero must be read when FILTER expects a measurement. Control then is returned to the main program and subsequently to INPUTF before returning to FILTER.

1. INPUT common data
 2. Blank or I6 zero to exit from INPUTF
 3. IX array card(s) to identify state variables and primary options
 - * 4. Measurement selector card
 - * 5. Considered parameters card
 - * 6. Measured elements and their epochs
 - ** 7. Blank or F10.2 zero to signal "data for next case following"
 - ** 8. INPUT common data for case 2
 - ** 9. Steps 2-7 for case 2, etc.
-
- * optional
- ** needed only if more than one case is to be run

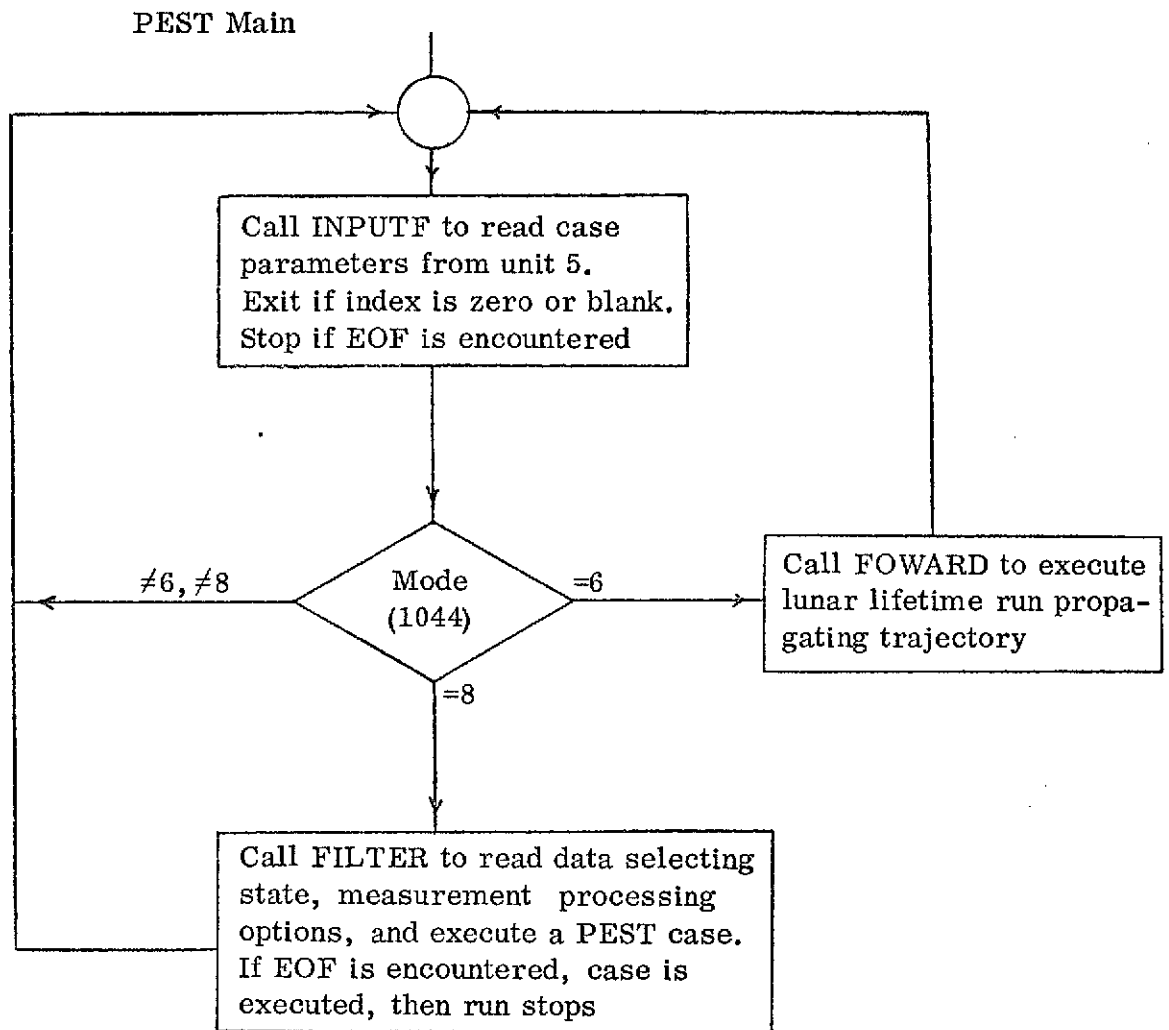


Figure 1.1 Input Data Read Logic

INPUT Common Data to Consider

General

1044	MODE	Flag to invoke PEST. Set to 8.
1030	KOUT	Output flag. Set to -1.
1033	KRW	Read-write flag for logical 22. Usually 1.
1049	KSAD	Shadow flag. Set to 0.

Trajectory Propagation

1036	KMETH	Propagation method flag. May be 4, 5, 7, or 8 unless KVAR = 1.
4	TFINAL	Final time (not used in estimation, where measurements control stop).
180	DELT	Integration step size (see KMETH).
1068	NINT	Number of intervals used in averaging.
1069	NORD	Number of ordinates used in averaging.
450	TOUT	Print table (set it larger than last measurement time)
460	STEP	Time interval for simulated measurements or when KRW is zero.

Field Model

1035	MODLEM	Lunar field model flag (3 for JPL 15, 8; 1 for L1).
1094	NMOD	Order of zonal harmonic model.
1095	MMOD	Order of tesseral harmonic model.
1084	KSOLP	Solar pressure flag (must be nonzero if SP is estimated).

Epoch

50-55		Input epoch to which all times are referred.
-------	--	--

Measurement Weights

56-76		Measurement weighting matrix (upper half of symmetric matrix).
-------	--	--

Initial Conditions

1019	KINPT	Input coordinate flag (see KAVST).
30-35	ELMI	Initial elements (see KAVST).
1098	KAVST	Averaging and initial condition flag. 0 neither input elements nor measurements are averaged. 1 average input elements if K METH is 5 or 8. 2 average measurements but not initial elements. 3 average both initial elements and measurements. -1 average nothing, but set initial elements to first measurement. -2 average measurements, set initial elements to first measurement.

Sensitivity Calculation

1096	KVAR	Sensitivity method flag. 0 for secant, 1 for variational.
1064	NGROPT	Number of trials for which sensitivities are to be recomputed.
1086	KZIP	Flag for ignoring implicit term in variational equations.

Specification of State Variables and Options

The IX array is read in by means of one or more records (cards) written in a 20I3 format. Reading of these records is terminated when the (N+1)st element of IX is negative or zero, or when five records (100 elements) have been read. Valid identification numbers are:

Field coefficients	IX(i) = 2-271 except MOD(IX(i),16) = 0 or when IX(i) lies outside the field specified by NMOD, MMOD.
Initial conditions	IX(i) = 273-278.
Solar pressure	IX(i) = 279, 280.

The first nonpositive identifier controls the operational logic:

IX(1)	= 0	Generate simulated measurement data.
IX(N+1)	= 0	Estimate the state if N is greater than zero.
IX(N+1)	= -1	Compute measurement residuals and (if N is greater than 0) compute/print measurement sensitivities without actually estimating a new state.

The field identifiers can be obtained with the assistance of Table 1.1. Find the appropriate coefficient in the table and add its row-number (shown on the right) to the number shown above its column. This sum is the identifier to use in the IX array. The order in which you specify these numbers is now immaterial — IX will be sorted appropriately by the program.

Examples:

__0	Simulated data will be generated from time zero to TFINAL by STEP.
__2__-1	Residuals and sensitivity to C_{20} will be printed.
_34274__0	Estimate C_{22} and either e_0 or $e_0 \cos \omega_0$, depending on KMETHOD.

Measurement Selector Card

The measurement selector (IM) card is read by means of a 2013 format. The first six words are the numbers (1-6) of the elements to be used in processing. A zero in the field will exclude the corresponding element. (1,2,3,0,5,6 excludes $\omega+f$ residuals if Method 7, $\omega+M$ if Method 8, is used, or ω if Methods 4 or 5 are used in propagation.) The next number is the number of observations to be processed. If this is greater than the number of observations to be found on tape, only the smaller number will be processed, of course. The next number dictates IQ, the weighting matrix treatment. If IQ is 1, Q is I; if IQ is 2, Q is constant; and if IQ is 3, Q is variable and is read in from the tape with the observation. In this latter case, the input weighting matrix is assumed to describe the uncertainty in the Cartesian state and will be converted appropriately by the program to an element weighting matrix. If Q is to be treated as a constant, it must be read in (locations 56 through 76) as an element weighting matrix, as it will not be converted by the program. In either case, Q must be positive definite.

ITMAX, the next indicator on the card, limits the number of iterations allowed in estimation if convergence is not achieved. The estimator will quit after passing through the data span ITMAX+1 times. The next indicator on the card is IGAIN,

Table 1.1
Field Identifier Aid

	0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	
	$c_{1,0}$	$c_{1,1}$	$s_{1,1}$	$s_{2,1}$	$s_{3,1}$	$s_{4,1}$	$s_{5,1}$	$s_{6,1}$	$s_{7,1}$	$s_{8,1}$	$s_{9,1}$	$s_{10,1}$	$s_{11,1}$	$s_{12,1}$	$s_{13,1}$	$s_{14,1}$	$s_{15,1}$	1
	$c_{2,0}$	$c_{2,1}$	$c_{2,2}$	$s_{2,2}$	$s_{3,2}$	$s_{4,2}$	$s_{5,2}$	$s_{6,2}$	$s_{7,2}$	$s_{8,2}$	$s_{9,2}$	$s_{10,2}$	$s_{11,2}$	$s_{12,2}$	$s_{13,2}$	$s_{14,2}$	$s_{15,2}$	2
	$c_{3,0}$	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$	$s_{3,3}$	$s_{4,3}$	$s_{5,3}$	$s_{6,3}$	$s_{7,3}$	$s_{8,3}$	$s_{9,3}$	$s_{10,3}$	$s_{11,3}$	$s_{12,3}$	$s_{13,3}$	$s_{14,3}$	$s_{15,3}$	3
	$c_{4,0}$	$c_{4,1}$	$c_{4,2}$	$c_{4,3}$	$c_{4,4}$	$s_{4,4}$	$s_{5,4}$	$s_{6,4}$	$s_{7,4}$	$s_{8,4}$	$s_{9,4}$	$s_{10,4}$	$s_{11,4}$	$s_{12,4}$	$s_{13,4}$	$s_{14,4}$	$s_{15,4}$	4
	$c_{5,0}$	$c_{5,1}$	$c_{5,2}$	$c_{5,3}$	$c_{5,4}$	$c_{5,5}$	$s_{5,5}$	$s_{6,5}$	$s_{7,5}$	$s_{8,5}$	$s_{9,5}$	$s_{10,5}$	$s_{11,5}$	$s_{12,5}$	$s_{13,5}$	$s_{14,5}$	$s_{15,5}$	5
	$c_{6,0}$	$c_{6,1}$	$c_{6,2}$	$c_{6,3}$	$c_{6,4}$	$c_{6,5}$	$c_{6,6}$	$s_{6,6}$	$s_{7,6}$	$s_{8,6}$	$s_{9,6}$	$s_{10,6}$	$s_{11,6}$	$s_{12,6}$	$s_{13,6}$	$s_{14,6}$	$s_{15,6}$	6
∞	$c_{7,0}$							$c_{7,7}$	$s_{7,7}$									7
	$c_{8,0}$								$c_{8,8}$	$s_{8,8}$								8
	$c_{9,0}$									$c_{9,9}$	$s_{9,9}$							9
	$c_{10,0}$										$c_{10,10}$	$s_{10,10}$						10
	$c_{11,0}$											$c_{11,11}$	$s_{11,11}$					11
	$c_{12,0}$												$c_{12,12}$	$s_{12,12}$				12
	$c_{13,0}$													$c_{13,13}$	$s_{13,13}$			13
	$c_{14,0}$														$c_{14,14}$	$s_{14,14}$		14
	$c_{15,0}$															$c_{15,15}$	$s_{15,15}$	15

the reciprocal gain factor on the incremental state to be added to the state between trials. Thus, if IGAIN = 2, only half of the estimated increment will be added. LTOL, the next number on the card, controls the tolerance on the iteration as follows: TOL = 1.D-LTOL. The next and last number is an indicator for the "consider" option. If ICON = 0, nothing happens in this regard. If ICON \neq 0, another card is expected to identify the "considered" parameters. The "considered" parameters are identified by the same code as the IX array.

Input of Measured Elements

The current means for input of measured elements is by means of DATA5. Each record contains 8 words written in an 8F10.2 format. These words are:

YYMMDD	Year, month, and day of epoch (710824 is 8/24/71)
HHMMSS	Hour, minute, second UMT (183025.5 is 18:30:25.5)
A	Semi-major axis (km)
E	Eccentricity
F	True Anomaly (deg)
W	Argument of pericenter (deg)
I	Inclination (deg)
O	Longitude of the ascending node (deg)

If KMETHOD is 5 or 8 and KAVST is -2, 2, or 3, the input measurements are considered to be osculating and selenographic and will be averaged and converted to mean of 1950.0 before being written onto logical 22. If KMETHOD is 5 or 8 but KAVST is -1, 0, or 1, the input measurements (and epochs) will be considered to be already averaged and converted to mean of 1950.0, so will be simply converted to KMETHOD variables before being written onto 22. If KMETHOD is 4 or 7 (KAVST should then be -1, 0, or 1), the input measurements will be considered to be osculating and selenographic and will be converted to KMETHOD variables, mean of 1950.0. The longitude parameter for KMETHOD = 8 is treated as $\omega+f$ outside the propagation routines when computing residuals and sensitivities.

Input Measurements from GTDS

The GTDS measurements are expected on magnetic tape in the form of

YYMMDD	Year, month, and day of epoch
HHMMSS	Hour, minute, and second of epoch
X Y Z · X · Y · Z	Cartesian position components (km), selenocentric, Equator of 1950.0
	Cartesian velocity components (km/sec), selenocentric, Equator of 1950.0

Definition of Peripheral Units

The parameter estimation program uses logical units 21, 22, and 23 as sequential datasets which must be defined in the JCL of the job setup. Logical 21 is used for input measurement data from GTDS. (Two words of epoch and six components of Cartesian state, EE50 in an 8D23.15 format, requiring a record length of 188 bytes.) Logical 22 is used for storage of the measured time and elements. It therefore requires a record length (LRECL) of 60 (8*7+4) or greater and a block-size of J*LRECL+4. Logical 23 is used as a scratch dataset for storage of the weighted sensitivities. It requires a record length of 48*N+4 or greater, where N is the number of state variables included in the case.

1.3 Input/Output Examples

Figure 1.2 shows the input which resulted in the output of Figure 1.3. The INPUTF data is abbreviated in Figure 1.2, but fully printed in Figure 1.3. The important numbers of the INPUTF data are:

IDM

FORTRAN Coding Form

PROGRAM		PUNCHING INSTRUCTIONS	GRAPHIC				
PROGRAMMER	DATE		PUNCH				

[illegible]

Figure 1.2 Estimation Input

DATA CARDS FOR CASE 1

30	0.2838000000	04	31	0.3957000000	02	32	0.1235210000	03
33	0.1046990000	03	34	0.1161580000	03	35	0.3639900000	02
50	0.6030000000	01	51	0.3000000000	02	52	0.1973000000	04
53	0.1500000000	02	54	0.1400000000	02	55	0.4300000000	02
1044	0.8000000000	01	1030	0.1000000000	01	1060	0.0	
4	0.3456000000	06	460	0.4320000000	05	450	0.1000000000	08
1035	0.3000000000	01	1094	0.4000000000	01	1095	0.4000000000	01
1098	0.0		0	0.0		0	0.0	
1036	0.8000000000	01	180	0.4320000000	05	1098	-0.2000000000	01
1036	0.7000000000	01	180	0.1200000000	04	1098	-0.1000000000	01
1096	0.0		0	0.0		0	0.0	
1096	0.1000000000	01	1086	0.1000000000	01	0	0.0	

INITIAL JULIAN DATE 2441864.1352

INITIAL CONDITIONS

IN EARTH MEAN EQUATOR AND EQUINOX OF 1950

6/30/73 15 HRS 14 MINS 43.000 SECS

X

Y

Z

-0.745050720 03 0.257752740 04 -0.943694220 03

XDOT

YDOT

ZDOT

0.945352200 00 -0.541860760-01 -0.907426940 00

ATTITUDE INPUT KEY (LOC 1087) = 0

CURRENT ATTITUDE, RA = -3.281 DEC = -43.780

IM M IQ IT IG LT

1	2	3	4	5	6	10	1	3	1	8
2			C(2, 0)		-0.1996E-03					2003
18			C(2, 1)		0.8171E-05					2019
49			S(2, 1)		-0.7213E-05					2319
34			C(2, 2)		0.2359E-04					2035
50			S(2, 2)		0.4638E-05					2335
3			C(3, 0)		-0.5878E-05					2004
19			C(3, 1)		0.3001E-04					2020
65			S(3, 1)		0.1421E-05					2320
35			C(3, 2)		0.4698E-05					2036
66			S(3, 2)		0.5748E-06					2336
51			C(3, 3)		0.4847E-05					2052
67			S(3, 3)		-0.2919E-05					2352
273				P						
276				W+M						
277				INC						
278				LAN						

PARAMETER ESTIMATION

TRIAL NO. 0

730614.	105641.	-1757.205	2072.151	-966.683	-0.59180	-0.51858	0.69234
730614.	225641.	-2047.909	1890.460	-741.893	-0.87024	-0.64524	0.74686
730615.	105641.	-2298.148	1670.808	-502.309	-0.73142	-0.75892	0.78677
730615.	225641.	-2503.002	1417.462	-252.365	-0.57812	-0.85729	0.81129
730616.	105641.	-2658.356	1135.349	3.288	-0.41343	-0.93837	0.81989
730616.	225641.	-2760.986	830.001	259.812	-0.24071	-1.00049	0.81238
730617.	105641.	-2808.672	507.483	512.230	-0.06356	-1.04244	0.78886
730617.	225641.	-2800.316	174.295	755.508	0.11430	-1.06343	0.74979
730618.	105641.	-2736.051	-162.767	984.684	0.28910	-1.06317	0.69601
730618.	225641.	-2617.298	-496.797	1195.024	0.45715	-1.04190	0.62873

INITIAL METHOD 7 ELEMENTS AT TIME =

0.

Figure 1.3 2937.66919 -0.00981 0.01592 5.71579 0.67350 2.73629
(cont.)

ELEMENT RESIDUALS						
TIME	P	ESW	ECW	W+F	I	O
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.500	-0.09775	-0.00001	-0.00004	-0.00133	-0.00022	0.00212
1.000	-0.18543	-0.00002	-0.00008	-0.00261	-0.00044	0.00421
1.500	-0.25638	-0.00005	-0.00011	-0.00384	-0.00066	0.00629
2.000	-0.30516	-0.00008	-0.00013	-0.00500	-0.00088	0.00834
2.500	-0.32801	-0.00012	-0.00014	-0.00610	-0.00110	0.01037
3.000	-0.32308	-0.00015	-0.00014	-0.00714	-0.00131	0.01238
3.500	-0.29068	-0.00018	-0.00012	-0.00811	-0.00152	0.01438
4.000	-0.23337	-0.00020	-0.00009	-0.00902	-0.00172	0.01635
4.500	-0.15578	-0.00022	-0.00006	-0.00986	-0.00192	0.01831

ESTIMATED STATE INCREMENT (10 OBS)

2	18	49	34	50	3
-0.3099D-03	-0.4416D-07	0.1416D-04	0.3965D-06	0.1090D-06	-0.1626D-04
19	65	35	66	51	67
-0.2650D-06	-0.3183D-05	-0.1322D-06	0.3413D-06	-0.6601D-07	-0.7851D-07
273	276	277	278		
-0.2148D-05	0.4724D-05	0.4283D-05	0.4659D-05		

PENALTY FUNCTION DY*(Q**-1)*DY = 0.5796D 00

TRIAL NO. 1

INITIAL METHOD 7 ELEMENTS AT TIME = 0.

2937.66918 -0.00981 0.01592 5.71579 0.67351 2.73630

ELEMENT RESIDUALS						
TIME	P	ESW	ECW	W+F	I	O
0.0	0.00000	0.0	0.0	-0.00000	-0.00000	-0.00000
0.500	-0.00030	0.00000	-0.00000	-0.00000	-0.00000	-0.00000
1.000	-0.00047	0.00000	-0.00000	0.00000	-0.00000	-0.00000
1.500	-0.00050	0.00000	-0.00000	0.00001	-0.00001	-0.00000
2.000	-0.00038	0.00000	-0.00000	0.00001	-0.00001	-0.00000
2.500	-0.00014	0.00000	-0.00000	0.00002	-0.00001	-0.00001
3.000	0.00018	0.00000	0.00000	0.00002	-0.00002	-0.00001
3.500	0.00052	0.00000	0.00000	0.00003	-0.00002	-0.00002
4.000	0.00079	-0.00000	0.00001	0.00003	-0.00003	-0.00003
4.500	0.00093	-0.00000	0.00001	0.00003	-0.00003	-0.00004

ESTIMATED STATE INCREMENT (10 OBS)

2	18	49	34	50	3
0.7981D-06	0.7920D-07	0.2209D-06	-0.3975D-06	-0.8704D-07	0.4625D-05
19	65	35	66	51	67
0.2844D-06	-0.3418D-06	-0.1278D-06	-0.3416D-06	0.7005D-07	-0.7837D-07
273	276	277	278		
0.2133D-05	-0.4716D-05	-0.4178D-05	-0.4560D-05		

PENALTY FUNCTION DY*(Q**-1)*DY = 0.2521D-05

TRIAL NO. 2

INITIAL METHOD 7 ELEMENTS AT TIME = 0.

2937.66919 -0.00981 0.01592 5.71579 0.67350 2.73630 18

Figure 1.3 (cont.)

ELEMENT RESIDUALS

TIME	P	ESW	ECW	W+F	I	O
0.0	0.00000	0.0	0.0	-0.00000	-0.00000	-0.00000
0.500	0.00000	-0.00000	0.00000	-0.00000	-0.00000	-0.00000
1.000	0.00000	-0.00000	0.00000	0.00000	-0.00000	-0.00000
1.500	0.00000	-0.00000	0.00000	0.00000	-0.00000	-0.00000
2.000	-0.00000	-0.00000	0.00000	0.00000	-0.00000	-0.00000
2.500	-0.00000	-0.00000	0.00000	0.00000	-0.00000	-0.00000
3.000	-0.00001	-0.00000	0.00000	0.00000	-0.00000	-0.00000
3.500	-0.00000	-0.00000	-0.00000	0.00000	-0.00000	-0.00000
4.000	0.00000	-0.00000	-0.00000	0.00000	-0.00000	-0.00000
4.500	0.00002	0.00000	-0.00000	0.00000	0.00000	-0.00000

ESTIMATED STATE INCREMENT (10 OBS)

2	18	49	34	50	3
-0.4709D-07	-0.3878D-07	0.4209D-07	-0.8002D-09	-0.1154D-07	-0.1162D-06
19	65	35	66	51	67
-0.1049D-07	-0.1504D-08	0.1209D-08	0.6469D-08	-0.2008D-08	-0.9500D-09
273	276	277	278		
0.7430D-08	0.2543D-07	-0.5584D-07	-0.8639D-07		

PENALTY FUNCTION DY*(Q**-1)*DY = 0.3357D-09

TRIAL NO. 3

INITIAL METHOD 7 ELEMENTS AT TIME = 0.

2937.66919 -0.00981 0.01592 5.71579 0.67350 2.73629

ELEMENT RESIDUALS

TIME	P	ESW	ECW	W+F	I	O
0.0	0.00000	0.0	0.0	-0.00000	-0.00000	-0.00000
0.500	0.00000	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000
1.000	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000
1.500	-0.00000	-0.00000	-0.00000	0.00000	-0.00000	-0.00000
2.000	0.00000	-0.00000	-0.00000	0.00000	0.00000	-0.00000
2.500	-0.00000	-0.00000	-0.00000	0.00000	0.00000	0.00000
3.000	-0.00000	-0.00000	-0.00000	0.00000	0.00000	0.00000
3.500	-0.00000	0.00000	-0.00000	0.00000	0.00000	0.00000
4.000	-0.00000	0.00000	-0.00000	-0.00000	0.00000	-0.00000
4.500	-0.00000	0.00000	-0.00000	-0.00000	0.00000	-0.00000

ESTIMATED STATE INCREMENT (10 OBS)

2	18	49	34	50	3
-0.1175D-08	-0.7891D-09	-0.2940D-10	0.3445D-09	-0.1252D-09	-0.1728D-09
19	65	35	66	51	67
-0.8934D-10	-0.5131D-09	0.1053D-09	0.1771D-09	-0.3633D-10	-0.9672D-10
273	276	277	278		
-0.3309D-08	0.2093D-08	0.4998D-08	0.4244D-08		

PENALTY FUNCTION DY*(Q**-1)*DY = 0.7622D-11

SQUARE-ROOT INFORMATION MATRIX

-0.2716D 04
 -0.2448D 04 -0.1954D 04
 -0.5719D 04 -0.3468D 04 -0.5491D 03
 0.2753D 04 -0.8143D 02 0.2543D 03 -0.6909D 03
 0.4751D 00 0.8130D 03 -0.5316D 03 0.1521D 03 -0.4747D 03
 -0.1622D 04 -0.1379D 04 -0.3397D 03 -0.3297D 02 -0.3283D 02 -0.9599D 02

0.91990 04	0.30900 04	0.46610 03	-0.25320 03	0.48420 02	0.82620 02
-0.10010 03					
0.55930 04	-0.12610 04	-0.52420 03	0.27250 03	0.31360 03	0.69150 02
-0.28170 02	-0.35500 03				
0.23690 05	0.14150 05	0.26030 04	-0.48790 02	0.12200 02	0.46040 03
-0.15470 03	-0.27180 03	-0.14280 03			
-0.21650 04	-0.84640 04	-0.19080 04	0.18970 03	0.34900 03	-0.30160 03
-0.63880 02	-0.52200 03	-0.51650 02	-0.15300 03		
0.28180 05	0.13980 05	0.46780 04	-0.70250 03	0.69890 03	0.95280 03
0.31050 02	-0.27200 03	-0.31820 03	-0.10990 03	-0.13110 03	
-0.22130 05	-0.28630 04	-0.12840 04	-0.55740 03	0.54590 03	-0.34340 03
-0.20890 03	-0.69490 03	-0.19800 02	-0.35430 03	0.65250 01	-0.12790 03
-0.29170 01	-0.27850 00	0.59550 00	0.33460 00	0.15580 00	0.11300 00
-0.46290 01	-0.43940 00	0.23380 01	-0.70050 02	0.24690 01	-0.15510 00
0.10300 01					
0.13600 01	0.14160 01	-0.17370 01	0.16130 01	0.78750 01	-0.49200 00
0.39160 00	-0.57430 00	0.53230 00	-0.33530 00	0.20760 00	-0.14140 00
-0.80840 01	0.13900 01				
-0.19200 00	-0.70480 01	-0.23270 00	-0.68940 00	-0.10520 01	0.18290 01
0.33500 00	-0.83390 00	0.37670 00	-0.49560 00	-0.78480 00	-0.53920 00
-0.80450 01	0.39590 00	0.16590 01			
0.50730 00	0.27060 01	0.90030 00	0.12990 01	-0.18980 01	-0.12790 00
-0.36210 01	0.95480 00	-0.52720 00	0.14210 00	0.49450 00	-0.69760 00
-0.64870 01	0.21320 00	0.31430 00	0.13240 01		
2	C(2, 0)	0.10960 03	2003		
18	C(2, 1)	0.81660 05	2019		
49	S(2, 1)	0.72060 05	2319		
34	C(2, 2)	0.23590 04	2035		
50	S(2, 2)	0.45480 05	2335		
3	C(3, 0)	0.58780 05	2004		
19	C(3, 1)	0.30020 04	2020		
65	S(3, 1)	-0.14220 05	2320		
35	C(3, 2)	0.46950 05	2036		
66	S(3, 2)	0.58110 06	2336		
51	C(3, 3)	0.48490 05	2052		
67	S(3, 3)	-0.29200 05	2352		
273	P				
276	W+M				
277	INC				
278	LAN				

Figure 1.3 (cont.)

1044	MODE is 8 to signal parameter estimation
1030	Output key is negative to suppress extra output
1060	Zero signals no print of input common
1035	3 signifies the JPL 15,8 field is estimated
1094	4 means the field is truncated at $n=4$
1095	4 means the field is truncated at $m=4$
1036	Method 7 propagation is used (Gauss' elements, no averaging)
180	The integration time step is 20 minutes
1098	Initial conditions (time and elements) are taken from the measurement tape when KAVST=-1
1096	KVAR is 1 to use linear variational equations for measure- ment sensitivities
1086	KZIP is 1, but has no significance when 1036 is 7

The zero in column 6 of Figure 1.2 causes a return from INPUTF and continuation to FILTER where the next card is read. This card is to select parameters to be estimated. For the case shown, these were all of the gravitational coefficients of a 3,3 field plus four initial elements. The significance of the numbers is indicated in the printout shown in Figure 1.3. The next card says that residuals in all six Method 7 elements are to be processed in the estimation. It also says that only the first 10 sets of elements on the measurement file are to be processed, that no weighting is to be done ($IQ=1$), that no more than three iterations are to take place, that unit gain is to be put on the included state increment, that the convergence tolerance is 10^{-8} , and that no parameters are to be "considered." The floating-point zero which makes up the last card shown in Figure 1.2 indicates that there is no card-input measurement data for this case.

Referring now to Figure 1.3 under the heading "Parameter Estimation," we see eight columns of data. These are the measurements. Each row contains the date in the first two columns (YYMMDD, HHMMSS), then three components each of position

and velocity (selenocentric, EE50). The data shown here were generated by the GTDS program and placed on unit 21 for processing by PEST. They represented osculating sets generated by perturbing certain elements of the JPL 15, 8 field truncated to 4, 4. It was the intent of this PEST run to determine which coefficients had been perturbed and by how much. The Method 7 initial conditions are printed, followed by the residuals in Method 7 elements. The residuals are in kilometers and radians, while time is in days from epoch. The estimated state increment tells how much change is to be made in the estimated parameters from the residuals just printed. The penalty function printed out next is $\tilde{Y} Q^{-1} \tilde{Y}$ for all of the observations. This is the function to be minimized by the (weighted) least squares procedure. The estimation process takes three trials to converge because the RMS state increment is greater than 10^{-8} for the first two trials. We may notice that the penalty function has reduced from .5796 to $.7622 \times 10^{-11}$ after three iterations. The square-root information matrix is printed out in its triangular form (folded by format after 6 rows). This matrix provides information on the observability of the various parameters being estimated. The post-estimation gravitational coefficient values are printed out finally. These indicate that $C(2, 0)$, $S(2, 1)$, $C(3, 0)$, and $S(3, 1)$ were perturbed (by comparing with the initially estimated field).

Although the initial orbital element estimates are not printed, their increments for the three trials add to zero (essentially), so we can assume that the residuals were entirely due to the four perturbed coefficients. The fact that the residuals (and penalty function) on the last trial were so near to zero indicates that no parameters other than those estimated were perturbed. The measurement sensitivities for this process were computed only on the first trial, then "frozen" for the remaining trials without corrupting convergence appreciably.

Figure 1.4 shows input for a two-case run. No output is shown for this run. The first case is a simulation of measurements. Initial elements are taken

IBM

FORTRAN Coding Form

PROGRAM		PUNCHING INSTRUCTIONS	GRAPHIC			
PROGRAMMER			DATE	PUNCH		

COMM.	STATEMENT NUMBER	CONT.	FORTRAN STATEMENT																																																									
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
			INPUT DATA																																																									
		0																																																										
	0																																																											
	0.																																																											
		0																																																										
	0																																																											
	2	18	49	-1																																																								
	1	2	3	4	5	6	10	1	3	1	8	0																																																

Figure 1.4 Input for Simulation and Residual Print

from locations 30-35 in INPUT common, then converted according to KMETHOD (1036) for propagation. The time step for measurements (location 460) must also be provided for this case. The first zero in column 6 causes escape from INPUTF. The zero in column 3 tells FILTER that a simulation is to be performed. The floating-point zero is to "end" measurement data input from cards and signal another case following. The next 16 zero is read by INPUTF at the beginning of case 2, causing a return to FILTER. Notice that more INPUT cards could have been read in before leaving INPUTF. FILTER reads the next card. The -1 in column 12 tells FILTER not to call WLS, but to compute and print residuals. The 2, 18, and 49 indicate that sensitivities of the measurements to $C(2,0)$, $C(2,1)$, and $S(2,1)$ are to be printed. The only necessary quantities on the measurement selector card for this case are M and IQ. The last zero, which signals "no more" measurement data cards, is omitted since this is the last case of the run.

II. PEST PROGRAM DESCRIPTION (Programmer's Guide)

This section supplements Reference 2 by describing the PEST program insofar as it differs from MAESTRO.

2.1 Program Structure

Figure 2.1 is an overview of the structure of PEST and a flow chart of the main program or driver. The input and initialization functions for trajectory propagation are identical to those of MAESTRO. The primary difference is that PEST includes a path for $\text{MODE} = 8$, the parameter estimation mode. A lunar lifetime mode (6) is retained from the MAESTRO capabilities in order to verify the characteristics of an estimated lunar field. For further elaboration of PEST's structure, see first the flow chart for FILTER (Section 2.3) and then that for WLS.

2.2 Subroutine Cross-Reference and COMMON Blocks Descriptions

Table 2.1 lists the subroutines which make up the PEST program and defines their functions or purposes. It does not include those subroutines which are in common with MAESTRO.

Figure 2.2 is a cross-reference map for the parameter estimation portion of PEST. For further details, see Figures 3.1 and 3.3 of Reference 2.

Table 2.2 lists the COMMON blocks referred to by PEST subroutines. Those COMMON blocks marked by asterisks are found only in PEST — not in MAESTRO. The PEST-only COMMON blocks are described in Table 2.3. The others are described in Reference 2.

PEST Main Program Flow Chart

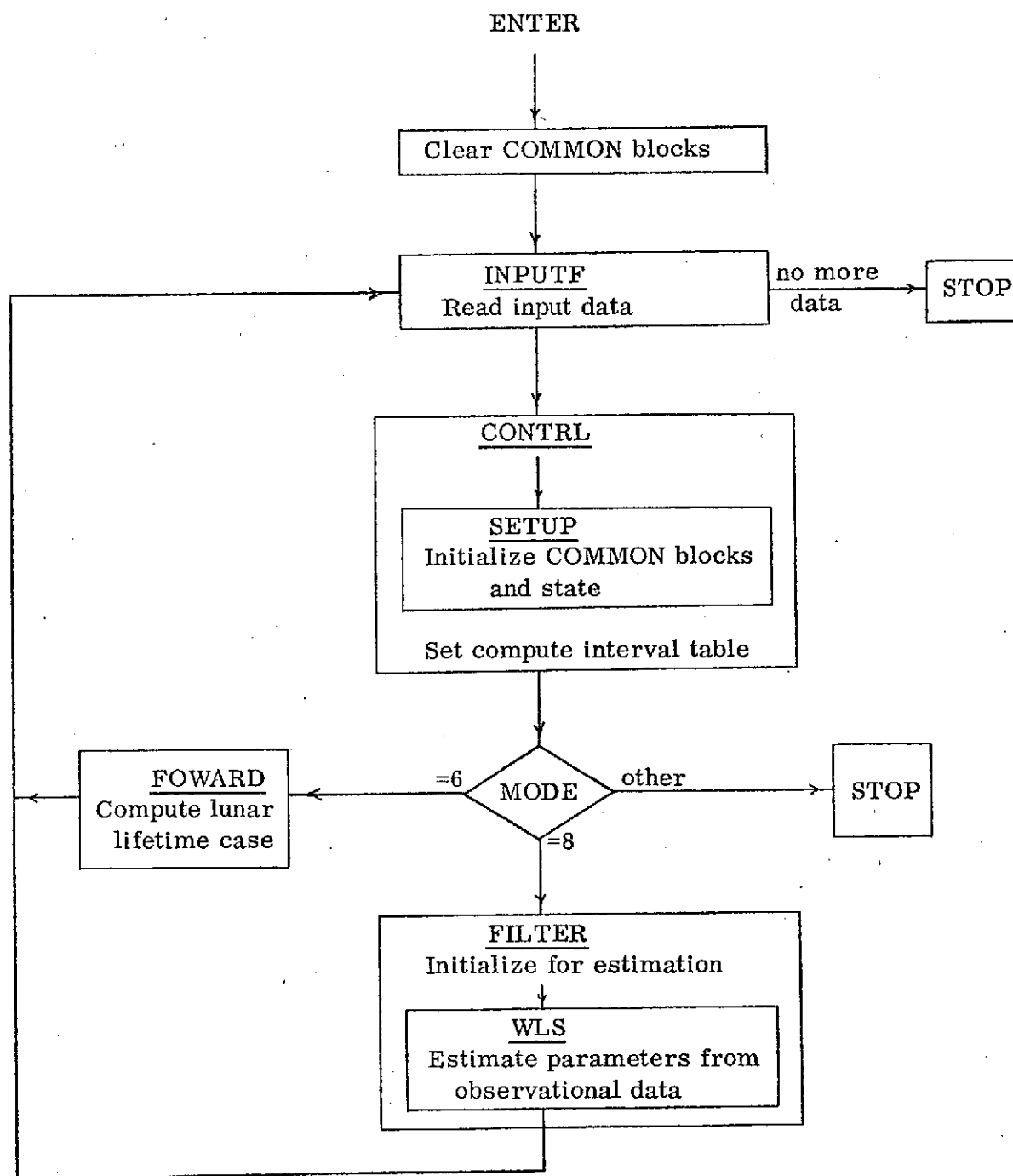


Figure 2.1 PEST Program Overview

TABLE 2.1

PEST SUBROUTINE DEFINITIONS

MAIN	Initiates and drives PEST
ACCEL	Determines spacecraft acceleration
AVEQNS	Averages derivatives of the elements over one revolution
AVSTRT	Averages osculating elements and variational equations
EQNS	Determines derivatives of the state for RKSEVN
FIELD2	Evaluates central aspherical disturbing acceleration and its gradient
FILTER	Initiates the parameter estimation process
GRAV	Evaluates disturbing acceleration from noncentral bodies
MATMPY	Matrix multiplication routine
MODEL	Controls state and sensitivity propagation
OSCAR	Averages osculating elements
OUTPUT	Prints trajectory and sensitivity information
PHOUSE	Performs matrix reduction by Householder's algorithm
PUTINV	Inverts a packed upper triangular matrix
REDATA	Initializes and reads measurement file
RKSEVN	Seventh-order Runge-Kutta integration routine
SHIMMY	Evaluates derivative for linear variational equations
SOLVE	Solves weighted least squares equation for state increment
SORDID	Sorts and prints parameters to be estimated
SPNM	Evaluates Legendre polynomials for FIELD2
SQRUT	Computes triangular square root of P-D symmetric matrix
TIMEC	Controls logic for trajectory propagation
TUMULT	Multiplies by a packed upper triangular matrix
UPDATX	Updates state with the estimated increment
WLS	Weighted least squares estimation routine

Figure 2.2 Parameter Estimation Cross-Reference Map

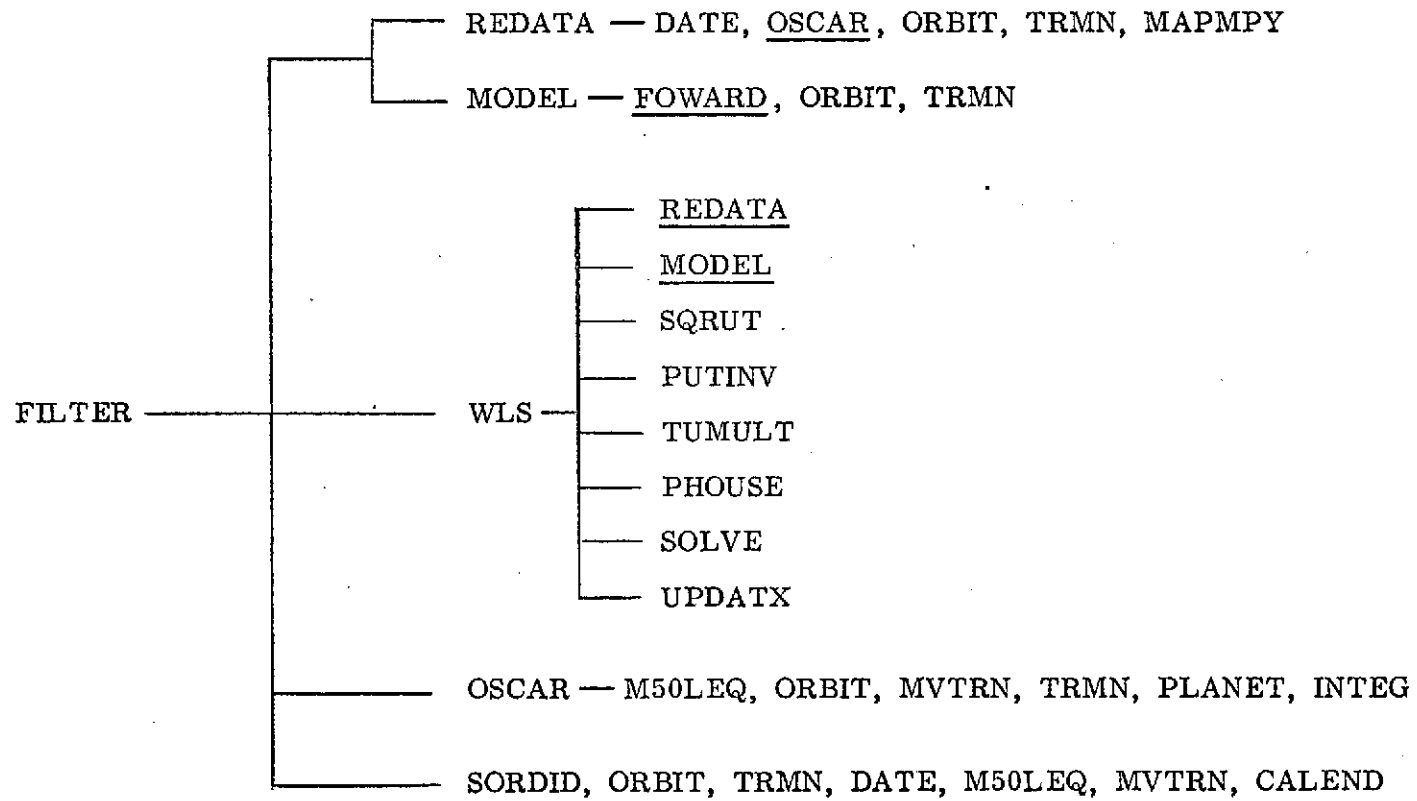


TABLE 2.2

COMMON BLOCKS

<u>COMMON</u>	<u>PEST Subroutines Referring to this COMMON</u>
BLANK	OUTPUT
AVG	MAIN, AVEQNS
CETBL2	MAIN
CETBL3	MAIN
*CNSDR	FILTER, UPDATX
CNTRL	MAIN, ACCEL, AVEQNS, AVSTRT, EQNS, FIELD2, GRAV, OUTPUT, RKSEVN, SHIMMY, TIMEC
*COMIC	REDATA, UPDATX
CONST	MAIN, AVEQNS, AVSTRT, EQNS, FIELD2, FILTER, GRAV, MODEL, OSCAR, OUTPUT, REDATA, SHIMMY, TIMEC, WLS
*DUMMY	MAIN, AVEQNS, EQNS, FIELD2, GRAV
*DXCOM	MODEL, PHOUSE, REDATA, SOLVE, UPDATX, WLS
FIELDM	MAIN, FIELD2, FILTER, MODEL, SHIMMY, SORDID, UPDATX
*FIELDX	ACCEL, FIELD2, GRAV, OUTPUT, SHIMMY
GRAVTY	ACCEL, AVEQNS, EQNS, FIELD2, GRAV, OUTPUT
*HELMO	FILTER, MODEL, REDATA
*IMCOM	FILTER, MODEL, REDATA, UPDATX, WLS
INPUT	MAIN, ACCEL, AVEQNS, AVSTRT, EQNS, FIELD2, FILTER, GRAV, MODEL, OSCAR, OUTPUT, REDATA, RKSEVN, SHIMMY, SORDID, TIMEC, UPDATX, WLS
INPUTS	MAIN
INTER	TIMEC
INTVAR	MAIN, AVEQNS, AVSTRT, EQNS, FIELD2, GRAV, OSCAR, OUTPUT, RKSEVN, SHIMMY, TIMEC
*INTVRX	MAIN, AVEQNS, AVSTRT, EQNS, FILTER, MODEL, OSCAR, OUTPUT, REDATA, RKSEVN, SHIMMY, TIMEC, WLS

*IXCOM	MAIN, AVSTRT, FIELD2, FILTER, MODEL, OUTPUT, PHOUSE, REDATA, SHIMMY, SOLVE, SORDID, UPDATX, WLS
PERT	MAIN, ACCEL, AVEQNS, EQNS, FIELD2, GRAV, SHIMMY, TIMEC
PLNET	MAIN, ACCEL, GRAV, OUTPUT
SAVE	MAIN, TIMEC
STATE	MAIN, AVEQNS, AVSTRT, EQNS, FIELD2, FILTER, MODEL, OSCAR, OUTPUT, REDATA, SORDID, TIMEC, UPDATX
*YCOM	FILTER, MODEL, REDATA, WLS

TABLE 2.3

PEST COMMON BLOCK DESCRIPTIONS

<u>Block</u>	<u>Length (DP words)</u>	<u>Use and Make-up</u>
COMIC	6	Used to transfer estimated initial elements ELMI(6) Estimated initial elements
DUMY	27	To communicate elements, transformations, rates DUM(6) Dummy element vector BUMROT(3, 3) Equatorial to orbital transformation ELMD(12) Storage for elements, sines, and cosines
DXCOM	5850	Communicates crucial estimation information DX(100) Estimated state increment HQY(100) Weighted state residual, $H^T \bar{Q}^{-\frac{1}{2}} \tilde{Y}$ QH(5650) Square-root information matrix and weighted sensitivities
FIELDX	309	transfers variational equation derivatives DVDX2(3, 3) Gradient of the disturbing acceleration DFDCX(3, 100) Explicit sensitivity of gravitational disturbing acceleration to estimated parameters
HELMO	12	Transfers estimated elements and secant increments ELMO(6) Current estimated orbital elements EODINK(6) Fraction of initial elements used in computing secant partials
IMCOM	4	Communicates measurement indicators, gain, tolerance IM(6) Measurement indicators IGAIN Reciprocal gain for including state increment LTOL Iteration convergence tolerance (negative \log_{10})
INTVRX	1212.5	Communicates measurement sensitivities and rates H(606) Measurement sensitivity, $\partial y / \partial x$ QATES(606) Rates of H for variational equations NQ Number of variational equations
IXCOM	54	Communicates estimation control indicators IX(100) Identification array for estimated parameters N Number of parameters to be estimated M1 Number of elements observed at one time M Number of observations (sets of size M1) NQH Number of elements in the square-root information matrix $N*(N+1)/2$

		IQ	Weighting treatment key
		ITMAX	Maximum iterations allowed
		ID	Observation counter ($1 \leq ID \leq M$)
		IT	Iteration counter ($1 \leq IT \leq ITMAX$)
YHAT	50		Communicates time, measurement, estimate, and weight
		TI	Current time
		YHAT(6)	Estimated elements
		TF	Next measurement time
		Y(6)	Measured elements
		Q(36)	Measurement weighting matrix

2.3 Subroutine Descriptions

Documentation is included here for those subroutines which are uniquely required by PEST. PEST requires many of the subroutines of MAESTRO, but since these are described in detail in Reference 2, only differences between MAESTRO and PEST versions of the common routines will be discussed. Table 2.4 categorizes the subroutines.

TABLE 2.4

PEST/MAESTRO SUBROUTINES

<u>PEST Subroutines</u>	<u>Modified MAESTRO Subroutines</u>
FILTER	MAIN
MODEL	ACCEL
OSCAR	AVEQNS
PHOUSE	AVSTRT
PUTINV	EQNS
REDATA	FIELD2
SHIMMY	GRAV
SOLVE	MATMPY
SORDID	OUTPUT
SQRUT	RKSEVN
TUMULT	SPNM
UPDATX	TIMEC
WLS	

The primary modification to the MAESTRO subroutines is to increase the size of INTVRX common from 13 to 1213 to permit calculation of 100 sets of linear variational equations. Other differences are noted below.

MAIN	FILTER is called when MODE = 8
ACCEL	Contains FIELDX common and initializes DVDX2 array for linear variational equations (LVE) computation

AVEQNS	Averages linear variational equations as well as elements
AVSTRT	Averages initial LVE as well as initial elements
EQNS	Calls SHIMMY in computation of LVE
FIELD2	Computes acceleration sensitivities to gravitational coefficients for LVE
GRAV	Computes gradient of non-central-body acceleration for LVE
MATMPY	Includes a $C=AB^T$ call (MATMPT)
OUTPUT	Includes extra output for LVE
RKSEVN	Has larger internal storage for INTVRX variables
SPNM	Generates Legendre polynomials a row-at-a-time for FIELD2
TIMEC	Only differences are for larger INTVRX common

SUBROUTINE FILTER

Calling Sequence: CALL FILTER

Purpose: FILTER initializes the estimation process and controls (1) simulated data generation and (2) nonestimation sensitivity and residual calculations

Common Blocks Required: CNSDR, CONST, FIELDM, HELMO, IMCOM, INPUT, INTVRX, IXCOR, STATE, YCOM

Subroutines Called: CALEND, DATE, MODEL, MVTRN, M50LEQ, ORBIT, OSCAR, REDATA, SORDID, TRMN, WLS

Input/Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	PI, PI2	1	CONST(2, 3)	$\pi, 2\pi$
I	GM	12	CONST(5)	Planetary gravitational constants
I	ZONL	288	FIELDM(1)	Lunar field coefficients
O	ELMO	6	HELMO(1)	Estimated orbital elements
O	H	606	INTVRX(1)	Measurement sensitivities
I	TFINAL	1	INPUT(4)	Final time
I	COV	21	INPUT(56)	Input weighting matrix
I	A(197)	2	INPUT(197)	Solar pressure coefficients
I	STEP	1	INPUT(460)	Simulated measurement time increment
O	KRASH	1	INPUT(1032)	Integration stop type key
I	KMETH	1	INPUT(1036)	Integration method key

I/O	Symbolic Name	Program Dimension	Common Block	Dimension
O	IX	100	IXCOM(1)	Estimated state identifiers
O	N	1	IXCOM(101)	Number of state components
O	MI	1	IXCOM(102)	Number of observables at each measurement
O	Y	6	YCOM(9)	Propagated elements
O	Q	36	YCOM(15)	Measurement weighting matrix
O	ICNSDR	20	CNSDR(1)	"Considered" parameter identifiers
I	DJO	1	INPUT(46)	Initial Julian date
I	KVAR	1	INPUT(1096)	Key for sensitivity generation method
I	KAVST	1	INPUT(1098)	Key for averaging elements, setting I.C.'s
O	NQ	1	INTVRX(1213)	Number of sensitivity elements
I	IM	6	IMCOM(1)	Measurement selector array
I	IGAIN	1	IMCOM(7)	State increment inclusion number
I	LTOL	1	IMCOM(8)	Estimation iteration tolerance number
O	ID	1	IXCOM(107)	Measurement counter (set for REDATA)
O	IT	1	IXCOM(108)	Iteration counter
O	EJO	1	STATE(26)	Ephemeris time epoch
O	M	1	IXCOM(103)	Number of measurements to be processed
O	NQH	1	IXCOM(104)	Number of elements in SRIM
O	IQ	1	IXCOM(105)	Weighting type key
O	ITMAX	1	IXCOM(106)	Maximum iterations in WLS
I	X	6	STATE(1)	Initial estimated Cartesian state
O	TI	1	YCOM(1)	Previous measurement time
O	TF	1	YCOM(8)	Current measurement time

Discussion

FILTER performs three basic functions

1. It initializes for WLS, the estimator
2. It contains logic for generating a simulated data file.
3. It contains logic for reading the data file, printing measurements, residuals and sensitivities.

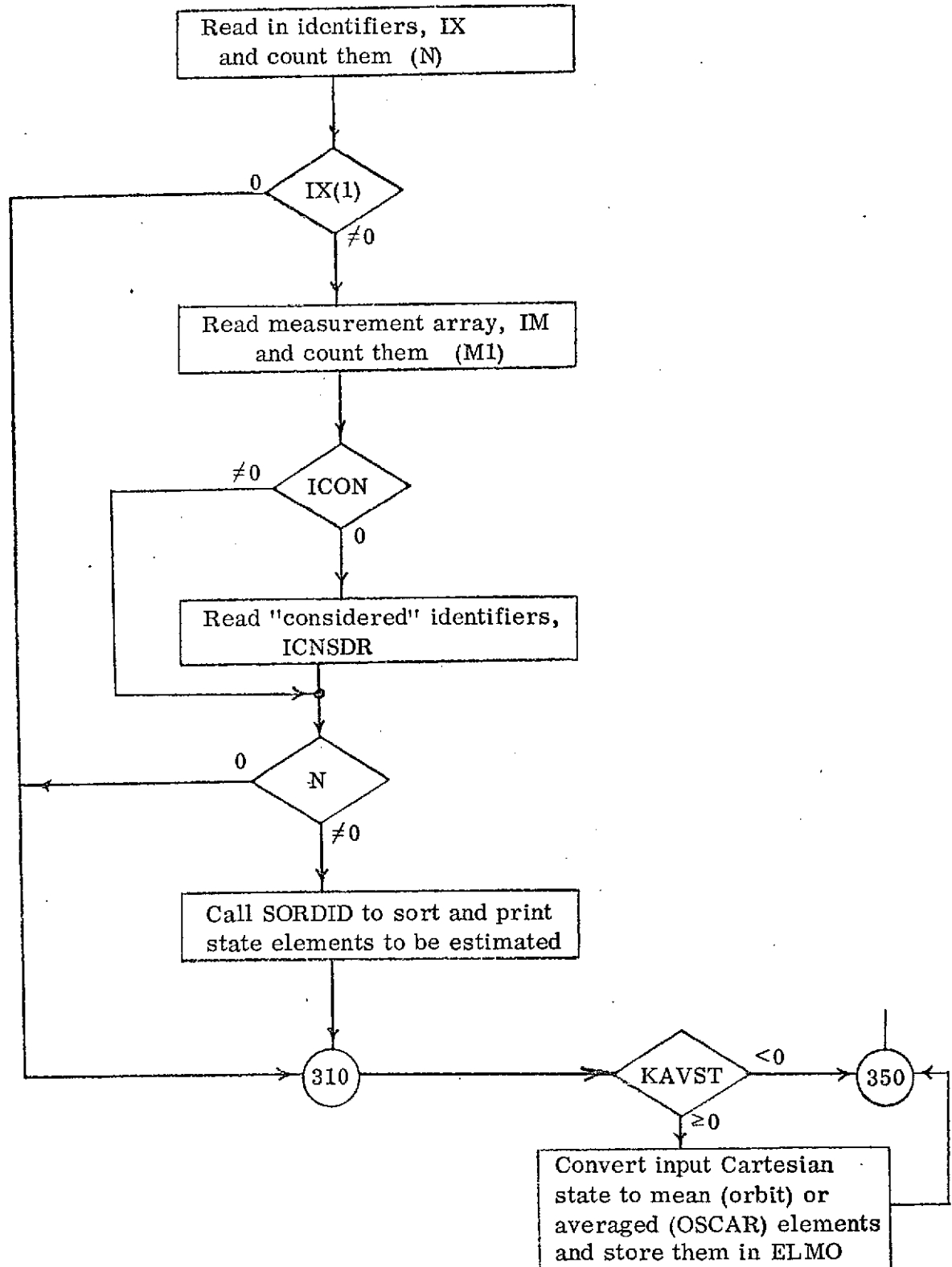
The IX array is read in from the input unit. It is this array which determines which function FILTER will perform:

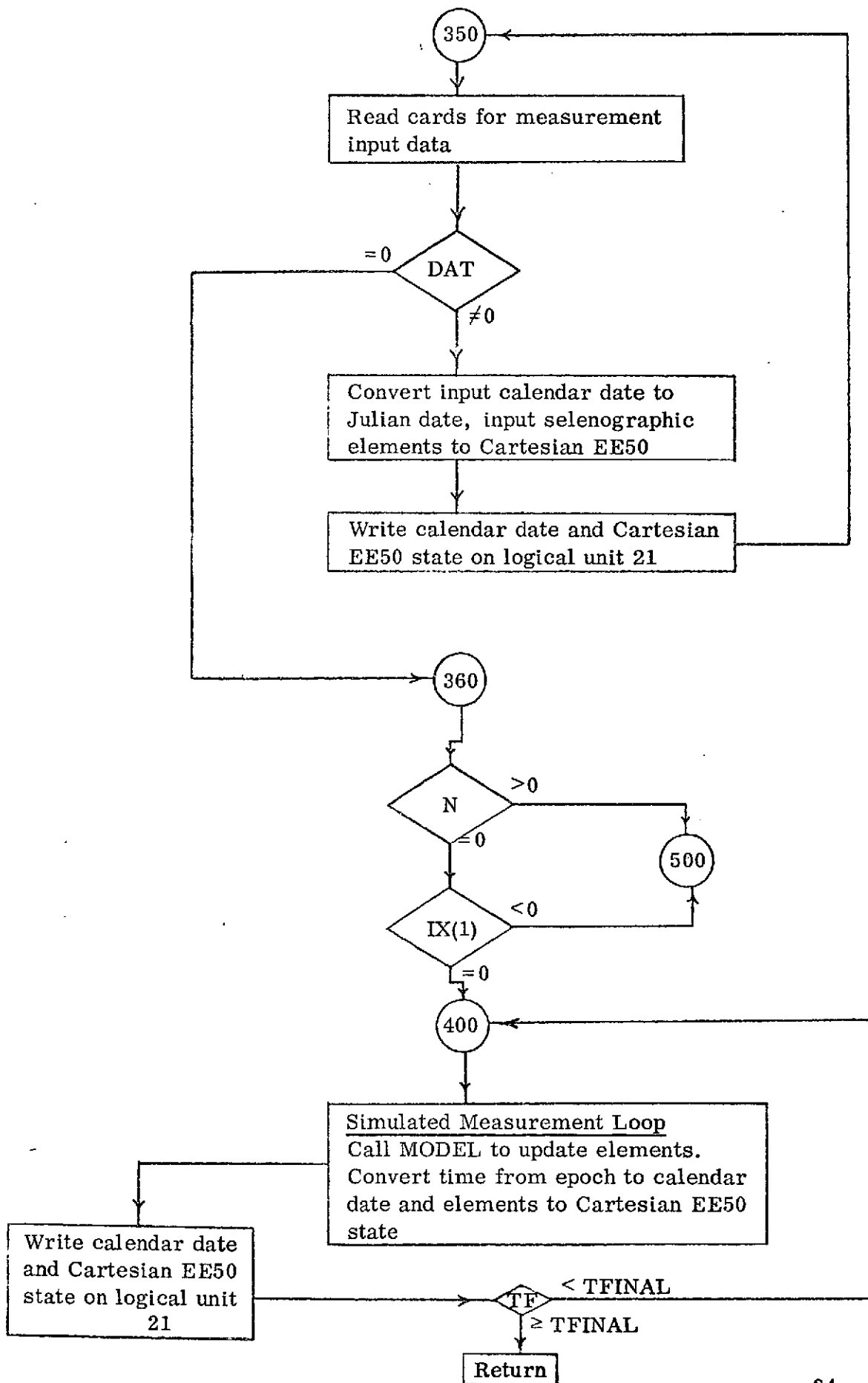
$IX(1) \leq 0$, $N = 0$, generate simulated data file

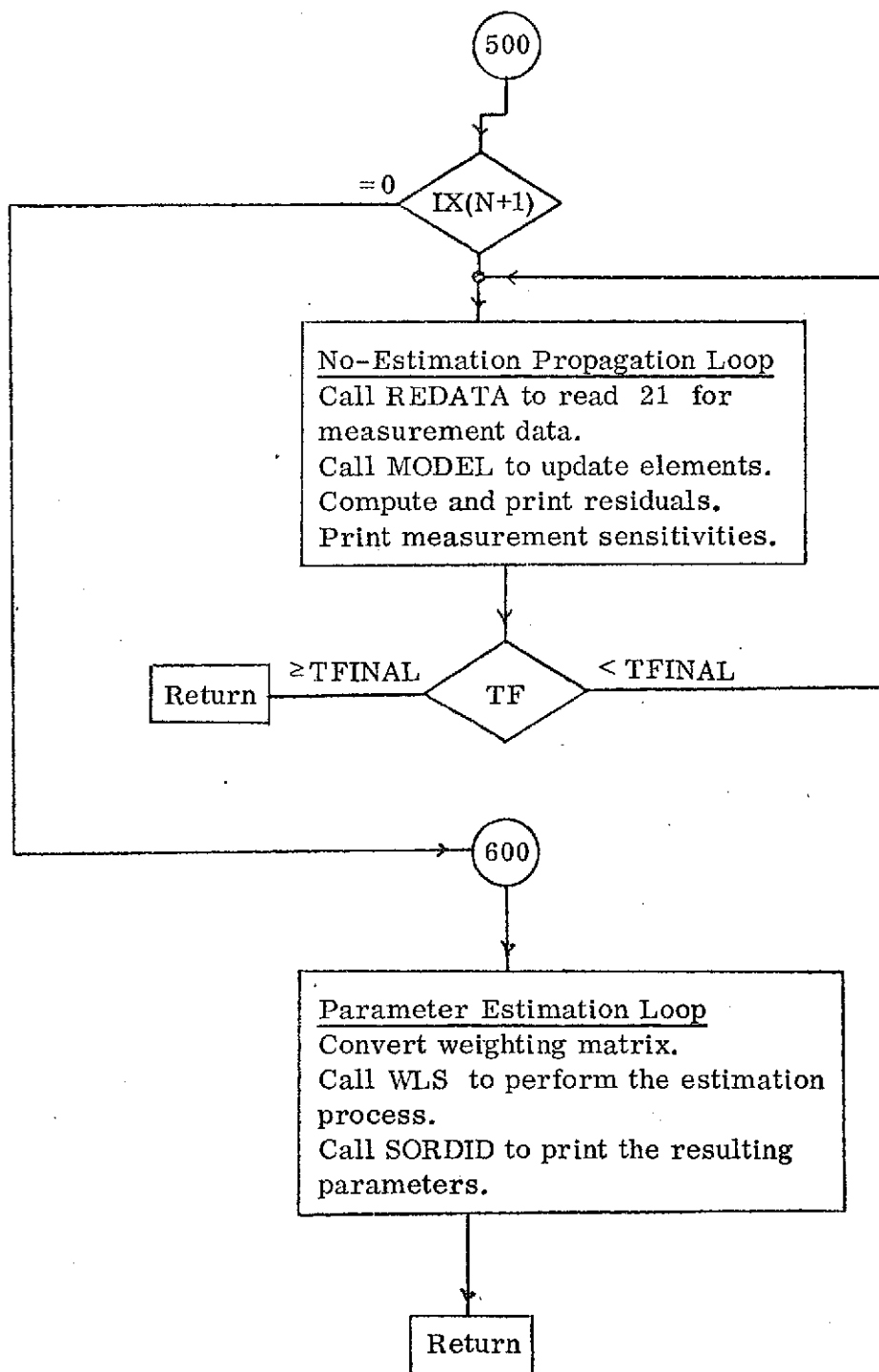
$IX(1) > 0$, $IX(N+1) = 0$, call the estimator, WLS

$IX(1) > 0$, $IX(N+1) < 0$, read data file, compute and print

FILTER FLOW CHART







SUBROUTINE MODEL

Calling Sequence:

CALL MODEL

Purpose:

MODEL computes the estimated measurement (orbital elements) and secant partials of the measurement with respect to state at a specified time.

Common Blocks Required:

CONST, DXCOM, FIELDM, HELMO, IMCOM, INPUT, INTVRX, IXCOS, STATE, YCOM

Subroutines Required:

FOWARD, ORBIT, TRMN

Input/Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	GM	12	CONST(5)	Gravitational constant array (GM)
O	QH	5650	DXCOM(201)	Measurement sensitivities to state
I/O	ZONL	16, 18	FIELDM(1)	Lunar field harmonic coefficients
I/O	ELMO	6	HELMO(1)	Estimated elements
I	EODINK	6	HELMO(7)	Initial element increments for partials
I/O	H	606	INTVRX(1)	Measurement sensitivities to state
I	NQ	1	INTVRX(1213)	Sensitivity computation key
O	TFINAL	1	INPUT(4)	Integration stop time
I/O	A123(198)	1	INPUT(198)	Solar radiation pressure coefficients
I	KMETH	1	INPUT(1036)	Integration method type
I	KVAR	1	INPUT(1096)	Sensitivity computation type key
I	IX	N	IXCOM(1)	State component identifier array
I	N	1	IXCOM(101)	Number of state components

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	NQH	1	IXCOM(104)	Number of square-root matrix elements
O	X	6	STATE(1)	Cartesian state of spacecraft
I	X(36)	1	STATE(36)	Solar pressure coefficient
O	T	1	STATE(10)	Time
O	ELM	6	STATE(14)	Orbital elements of spacecraft
I	TI	1	YCOM(1)	Entry time
O	YHAT	6	YCOM(2)	Estimated measurement (elements)
I	TF	1	YCOM(8)	Exit or measurement time
I	IM	6	IMCOM(1)	Measurement identifiers

Discussion

MODEL calls FOWARD to propagate the estimated orbital elements, ELMO, from TI to TF. Secant partials of the estimated measurement, YHAT = ELMO, are computed in H and in QH beginning with QH(NQH+1). The reason for double storage is that the QH array is destroyed by the estimation process in subroutine PHOUSE, while H is used to store initial conditions for integration between measurements. That is

$$H_{ij} = \begin{cases} \frac{\partial YHAT_i(T)}{\partial C_j} & \text{for gravitational or solar pressure coefficients, } C_j \\ \frac{\partial YHAT_i(T)}{\partial ELMO_j(0)} & \text{for initial condition states, } ELMO_j(0) \end{cases}$$

and is computed from

$$H(TF) = \frac{YHAT(TF, \hat{S} + \Delta) - YHAT(TF, \hat{S})}{\Delta} = \frac{YHAT(TF, \hat{S} + \Delta) - ELMO(TF)}{\Delta}$$

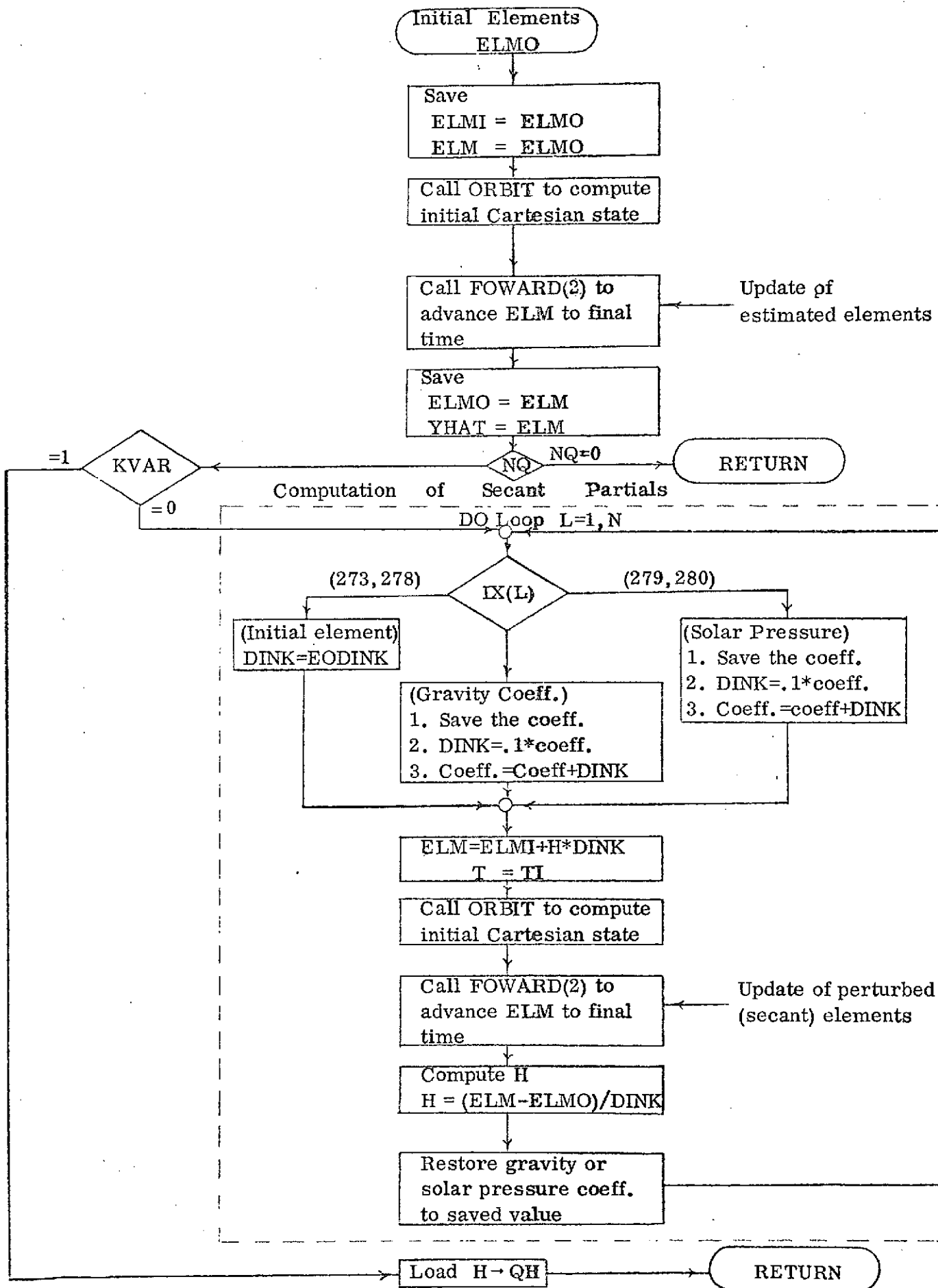
where \hat{S} is the estimated state and Δ is a small change in \hat{S} . Rather than carrying along $YHAT(T, \hat{S} + \Delta)$ from measurement to measurement, $YHAT(TI, \hat{S} + \Delta)$ is reconstructed from H to initiate integration over the next interval.

$$YHAT(TI, \hat{S} + \Delta) = ELMO(TI) + H(TI) * \Delta$$

If KVAR is nonzero, linear variational equations replace secant partials as the means for computing measurement sensitivities. These sensitivities reside in H (INTVRX) upon return from FOWARD.

The QH array (DXCOM) is upper-loaded from H according to the measurement identification array, IM. H is viewed as a $6 \times N$ array, while QH is viewed as an $M1 \times 6$ array, although both are programmed as single-dimensioned arrays.

MODEL Flow Chart



SUBROUTINE OSCAR

Calling Sequence: CALL OSCAR(KIN, TI, EI, TO, EO)

Purpose: OSCAR converts osculating input (elements or Cartesian state) to averaged elements.

Common Blocks Required: CONST, INPUT, INTVAR, INTVRX, STATE

Subroutines Called: INTEG, MVTRN, M50LEQ, ORBIT, PLANET

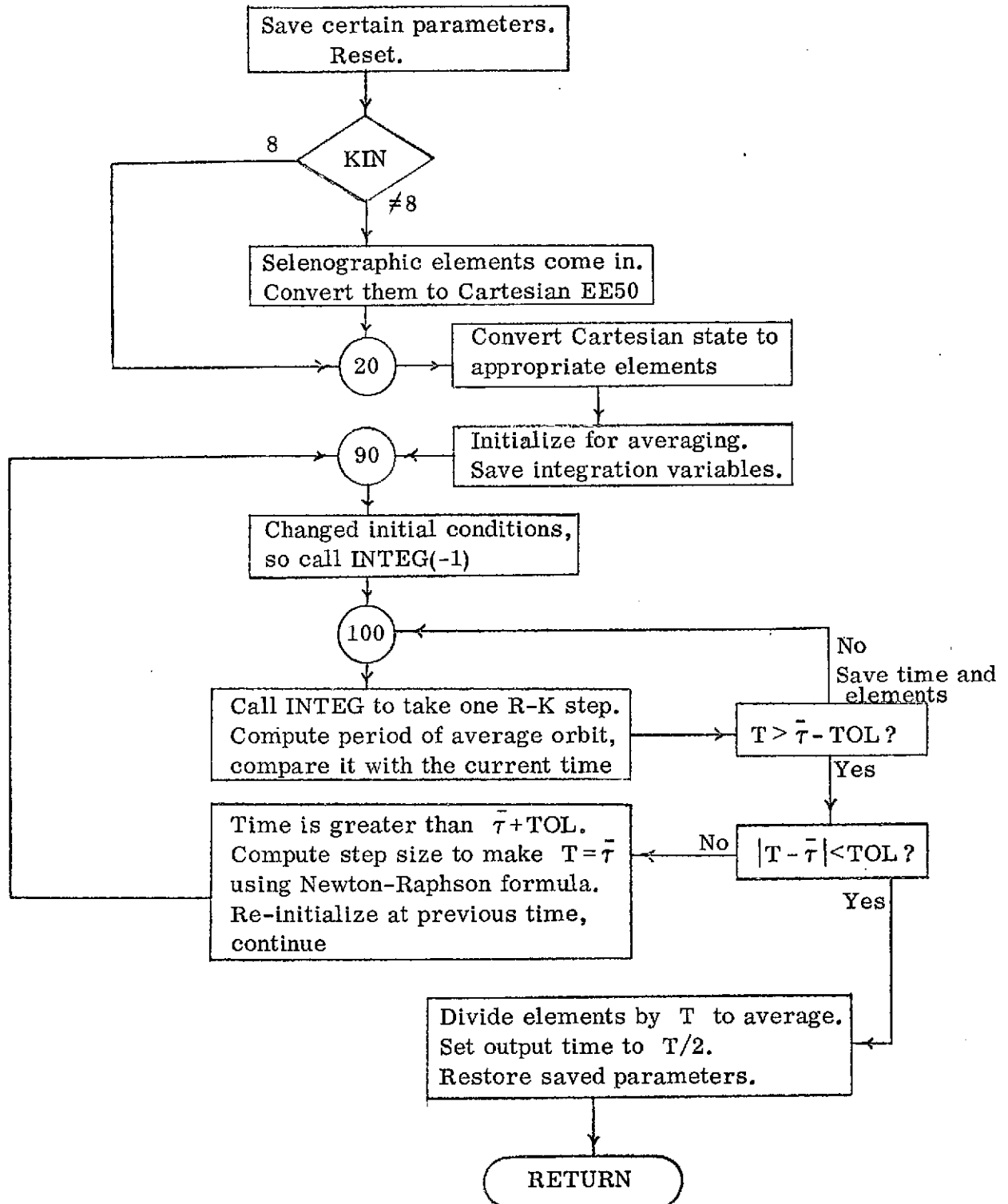
Input/Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	KIN	1	Call LIST	Input type key (8: Cartesian; other: elements)
I	TI	1	Call LIST	Osculating reference time from epoch
I	EI	6	Call LIST	Input osculating elements
O	TO	1	Call LIST	Output reference time from epoch
O	EO	6	Call LIST	Averaged elements output
I	RAD, PI2	1	CONST(1)	Angular constants
I	GM	12	CONST(5)	Gravitational constants
I	DJO	1	INPUT(46)	Julian date at epoch
I	KMETH	1	INPUT(1036)	Trajectory propagation method key
I	XX	6	STATE(1)	Input osculating Cartesian state
I/O	X, H	1	INTVAR(1)	Integration time and step
I/O	Z	6	INTVRX(1)	Intermediate osculating elements

Discussion

OSCAR is very similar to AVSTRT, but may be called at times other than epoch to average osculating sets other than the estimate. It is used to average "measured" osculating sets for use with averaged propagation methods in estimation. The osculating elements are integrated forward over one mean orbital period (found by iteration). The integral is then divided by that period to render the averaged elements, which are taken to be referred to the center of the integration interval.

OSCAR



SUBROUTINE PHOUSE

Calling Sequence:

CALL PHOUSE

Purpose:

PHOUSE reduces a partly-upper-triangular matrix to purely-upper-triangular, using Householder's Reduction Algorithm.

Common Blocks Required:

DXCOM, IXCUM

Subroutines Called:

None

Input / Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	N	1	IXCOM(101)	Columns of input and output matrix
I	MI	1	IXCOM(102)	Rows in excess of N
I	NQH	1	IXCOM(104)	$N * (N+1)/2$, non-zero elements of $N \times N$
I/O	QH, WT	$NQH+N*M1$	DXCOM(1)	Square-root information matrix (packed upper-triangular)

Limitations: QH is singly-dimensioned as follows

$$QH = (\underset{\substack{\uparrow \\ (NQH+N*M1)\text{-th} \\ \text{element}}}{h_{m,n}}, \underset{\substack{\uparrow \\ NQH\text{-th} \\ \text{element}}}{0} \rightarrow)$$

$w_{11}, w_{12}, w_{22}, w_{13}, w_{23}, \dots, w_{nn}, h_{11}, h_{21}, \dots, h_{m1}, h_{12}, h_{22}, \dots$

The input QH array is replaced by the output array. M1 must not exceed 6 and $NQH+N*M1$ must not exceed 105 unless program dimensions are changed.

Method

PHOUSE is a special-purpose implementation of Householder's Reduction Algorithm (1,2) for reducing a matrix to upper-triangular form. It is used in the weighted least squares

process to accumulate the information matrix, $H^T Q^{-1} H$, in upper-triangular square-root form. In this discussion, we can let $Q^{-1} = I$ without loss of generality.

Let us assume that at some point in a WLS process we have an upper-triangular SRIM (square-root information matrix), W , which is $N \times N$. That is, W contains $N \times (N+1)/2$ potentially non-zero elements above the diagonal and zeros below. W also contains all of the information from $k-1$ measurements.

$$W_{k-1}^T W_{k-1} = \sum_{i=1}^{k-1} H_i^T H_i \quad (1)$$

In the equation, H_i is the sensitivity of measurement to state at the i -th measurement. We now consider what happens to W when we add another measurement, whose sensitivity matrix is H_k . The new SRIM, W_k , must obey

$$W_k^T W_k = W_{k-1}^T W_{k-1} + H_k^T H_k \quad (2)$$

which could be written

$$\begin{array}{ccc} W_k^T W_k & = & \begin{bmatrix} W_{k-1}^T & H_k^T \end{bmatrix} \begin{bmatrix} W_{k-1} \\ H_k \end{bmatrix} \\ (nxn)(nxn) & & \begin{array}{cc} nx(n+m) & (n+m)xn \end{array} \end{array} \quad (3)$$

We may look at the reduction problem then as the problem of finding an orthogonal matrix, R , such that

$$\begin{bmatrix} W_k \\ 0 \end{bmatrix} = R \begin{bmatrix} W_{k-1} \\ H_k \end{bmatrix} \quad (4)$$

because

$$\begin{bmatrix} W_k \\ 0 \end{bmatrix}^T \begin{bmatrix} W_k \\ 0 \end{bmatrix} = \begin{bmatrix} W_{k-1} \\ H_k \end{bmatrix}^T R^T R \begin{bmatrix} W_{k-1} \\ H_k \end{bmatrix} \quad (5)$$

is identical to equation (3) for orthogonal R . Let us now simplify the notation by dropping the subscripts and renaming so that equation (4) becomes (6).

$$W = RA \quad (6)$$

Householder's choice of R is the product of a sequence of reflection matrices (which are symmetric and orthogonal).

$$R = R^{(n)} R^{(n-1)} \dots R^{(1)} \quad (7)$$

(Introduction of new indices is unavoidable)

$R^{(k)}$ is defined by

$$R^{(k)} = I - \beta_k W^{(k)} W^{(k)T} \quad (8)$$

where I is the $(n+m)$ identity, $W^{(k)}$ is a column vector to be developed, and

$$\beta_k = 2 / |W^{(k)}|^2. \quad (9)$$

The derivation of $W^{(k)}$ and $R^{(k)}$ is inductive. If A of equation (6) is partitioned by columns, a_i , as in equation (10)

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \quad (10)$$

we note from equation (2) that the first element of W must be $\pm a_1$. We define

$$\alpha_1 = |a_1| \text{ sign } a_1(1). \quad (11)$$

If we choose $R^{(1)}$ to zero the subdiagonal elements of $W^{(1)}$, where $W^{(k)}$ is defined by equation (12),

$$W^{(k)} = R^{(k)} R^{(k-1)} \dots R^{(1)} A \equiv R^{(k)} W^{(k-1)} \quad (12)$$

we may solve for $W^{(k)}$ (which turns out to be the k -th column of $W^{(k)}$).

$$R^{(1)} a_1 = a_1 \beta_1 W^{(1)} W^{(1)T} a_1 = \alpha_1 e_1 \quad (13)$$

In equation (13), e_1 is the first unit basis vector. The unit basis vector e_i has 1 as its i -th component and zeroes elsewhere. We may solve (13) for $W^{(1)}$,

$$W^{(1)} = \frac{a_1 + \alpha_1 e_1}{\beta_1 a_1^T W^{(1)}}, \quad (14)$$

but we do not know how to compute the denominator of (14). The length of $W^{(k)}$ in equation (8) is arbitrary, however, due to the definition of β_k , so we can simply define its length to be such that $\beta_1 a_1^T W^{(1)} = 1$. Then we have equation (15).

$$W^{(1)} \triangleq a_1 + \alpha_1 e_1 . \quad (15)$$

Having now solved for $W^{(1)}$, we could form $W^{(1)}$ from equation (12) by making the association, $W^{(0)} = A$. The first column of $W^{(1)}$ is zero, except for the first element, which is equal to $-\alpha_1$. We could now repeat the process, equation (12), by looking for $R^{(2)}$ which would zero the subdiagonal elements of the second column of $W^{(2)}$ without disturbing the first column. We can preserve the first column of $W^{(1)}$ under the transformation, $R^{(2)}$, by setting the first element of $W^{(2)}$ to zero. Then we have

$$R^{(2)} W^{(1)} = W^{(1)} - \beta_2 W^{(2)} \underbrace{W^{(2)T} W^{(1)}}_0 = W^{(1)} \quad (16)$$

The construction of $W^{(2)}$ is otherwise identical to that of $W^{(1)}$. Proceeding inductively, we can arrive at the following set of equations.

$$\alpha_k = \sqrt{\sum_{i=k}^m [W_{ik}^{(k)}]^2} \cdot \text{sign} [W_{kk}^{(k)}] \quad (17)$$

$$\beta_k = \frac{1}{\alpha_k (\alpha_k + W_{kk}^{(k)})} \quad (18)$$

$$W_i^{(k)} = \begin{cases} 0 & ; \quad i < k \\ \alpha_k + W_{kk}^{(k)} & ; \quad i = k \\ W_{ik}^{(k)} & ; \quad i = k+1, n+m \end{cases} \quad (19)$$

$$Y_i^{(k)} = \begin{cases} 0 & ; \quad i < k \\ 1 & ; \quad i = k \\ \beta_k W^{(k)T} W_i^{(k)} & ; \quad i = k+1, n+m \end{cases} \quad (20)$$

$W_i^{(k)}$ in equations (20) and (21) is the i -th column of $W^{(k)}$.

$$W_i^{(k+1)} = W_i^{(k)} - Y_i^{(k)} W^{(k)} \quad ; \quad i = k+1, n+m \quad (21)$$

Congratulations to you if you have followed the development to this point. It was not easy to write or type it, either.

In coding this routine, some advantage has been taken of the fact that the initial matrix is partly upper triangular at entry. That is, W_{k-1} of equation (4) is upper-triangular to begin with, so the dimension of $W_i^{(k)}$ in equation (19) needs only to be $m+1$ and the inner product of equation (20) needs to sum only m multiplications. These advantages can be significant if n is large.

References

1. Kaminski, P.G., "Square Root Filtering and Smoothing for Discrete Processes," SUDAAR No. 427, Guidance and Control Laboratory, Stanford University, Stanford, California, July 1971.
2. Householder, A.S., The Theory of Matrices in Numerical Analysis, Blaisdell Press, 1964, Chapter 5.

SUBROUTINE PUTINV

Calling Sequence: CALL PUTINV (B, C, N)

Purpose: PUTINV inverts a packed upper triangular matrix.

Common Blocks Required: None

Subroutines Called: None

Input/Output

I/O	Symbolic Name	Program Dimension	Communication	Definition
I	B	$N(N+1)/2$	Call List	Input matrix (P.U.T.)
O	C	$N(N+1)/2$	Call List	Output matrix (inverse of B, P.U.T.)
I	N	I	Call List	Rows or columns of upper triangular matrix

Note: B and C are single-dimensioned arrays containing the potentially non-zero elements of an upper triangular matrix.

$$B = (b_{11}, b_{12}, b_{22}, b_{13}, \dots, b_{nn})$$

Method

The input matrix, B, and its inverse, C, are treated as upper triangular arrays.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \text{etc.} \\ 0 & b_{22} & b_{23} & \text{etc.} \\ 0 & 0 & b_{33} & \text{etc.} \\ & \text{etc.} & & \end{bmatrix}, \quad \text{same for C.}$$

They must obey $CB = I$, which is equivalent to the following set of equations.

<u>$CB = I$</u>	<u>Solution</u>
$c_{11} b_{11} = 1$	$c_{11} = 1/b_{11}$
$c_{22} b_{22} = 1$	$c_{22} = 1/b_{22}$
$c_{11} b_{12} + c_{12} b_{22} = 0$	$c_{12} = -c_{22} c_{11} b_{12}$
$c_{33} b_{33} = 1$	$c_{33} = 1/b_{33}$
$c_{11} b_{13} + c_{12} b_{23} + c_{13} b_{33} = 0$	$c_{13} = -c_{33} (c_{11} b_{13} + c_{12} b_{23})$
$c_{22} b_{23} + c_{23} b_{33} = 0$	$c_{23} = -c_{33} (c_{22} b_{23})$
etc.	etc.
$\sum_{j=1}^n c_{ij} b_{jk} = \delta_{ik}$	$c_{kk} = 1/b_{kk}$
	$c_{jk} = -c_{kk} \left(\sum_{i=j}^{k-1} c_{ji} b_{ik} \right)$

B and C are treated as single - dimensioned arrays, so the programming requires calculation of the appropriate single indices which correspond to the double indices used in the development.

SUBROUTINE REDATA

Calling Sequence:

CALL REDATA

Purpose:

REDATA reads the data file and loads measurements into common. It also initializes certain variables at the beginning of the case and at the beginning of each iteration.

Common Blocks Required:

COMIC, CONST, DXCOM, HELMO, IMCOM, INPUT, INTVRX, IXCOR, STATE, YCOM

Subroutines Called:

DATE, MATMPT, MATMPY, ORBIT, OSCAR, TRMN

Input/Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I/O	ELMI	6	COMIC(1)	Estimated initial elements
O	QH	5650	DXCOM(201)	Initial sensitivity matrix for WLS
I/O	ELMO	6	HELMO(1)	Initial elements for integrator
O	EODINK	6	HELMO(7)	Initial element increments for secant partials
O	H	606	INTVRX(1)	Initial sensitivity matrix for MODEL
I	N	1	IXCOM(101)	Number of states
I	M1	1	IXCOM(102)	Number of observables at each measurement
O	M	1	IXCOM(104)	Number of measurements
I	NQH	1	IXCOM(105)	Number of elements in SRIM, $N(N+1)/2$
I	IQ	1	IXCOM(106)	Weighting flags
I	ID	1	IXCOM(108)	Measurement counter
I	IT	1	IXCOM(109)	Iteration counter
O	TI	1	YCOM(1)	Most recent measurement time

I/O	Symbolic Name	Program Dimension	Common Block	Definition
O	YHAT	6	YCOM(2)	Estimated measurement vector
I	GM	12	CONST(5)	Gravitational constants
I	DJO	1	INPUT(46)	Julian date at epoch (UMT)
I	KMETH	1	INPUT(1036)	Trajectory propagation method key
I	KAVST	1	INPUT(1098)	Averaging indicator, start-up key
O	EJO	1	STATE(36)	Ephemeris date at epoch
I/O	X	6	STATE(1)	Cartesian state
I/O	TF	1	YCOM(8)	Measurement time
O	Y	6	YCOM(9)	Measurement vector
O	Q	36	YCOM(15)	Measurement weighting matrix
I	IM	6	IMCOM(1)	Measurement identifiers

Discussion

Subroutine REDATA performs two functions: (1) loading the measurement from the data file and (2) initialization of certain program variables. The data file (logical 22) is read once on each entry with an unformatted binary read, expecting a time and six elements on each record. When an end-of-file is encountered, M, the number to signal "no more data — estimate the state now" is sent to WLS. The initialization functions are:

1. Load the estimated initial element vector with elements developed from inputs at the beginning of the case.
2. Load the integrator's initial element vector, ELMO, and the estimated initial measurement vector, YHAT, with the estimated initial elements at the beginning of each iteration or pass through the data.
3. Rewind the logical unit on which the data reside at the beginning of each iteration.

4. At the beginning of each iteration, set the measurement sensitivity matrix (H and part of QH) to zero -- or to one in the case of the sensitivity of an initial element measurement to an initial element state. The secant increment is also set when an initial element is to be estimated. The increment is 1/100,000 of the element's initial value.

Initialization Logic (see flow chart)

Input measurement data is assumed to reside on unit 21 in the form of calendar date plus Cartesian state (EE50). The state may be regarded as osculating or averaged. It will be converted to processing form depending upon the propagation method, KMETHOD, and the averaging key, KAVST. The time from epoch and the converted state (elements of the KMETHOD type) are written onto logical unit 22. If KMETHOD is 4 or 7, the measurements need be converted only once, but if KMETHOD is 5 or 8 (averaging), the measurements will be re-averaged at each iteration with the current estimated lunar gravitational field.

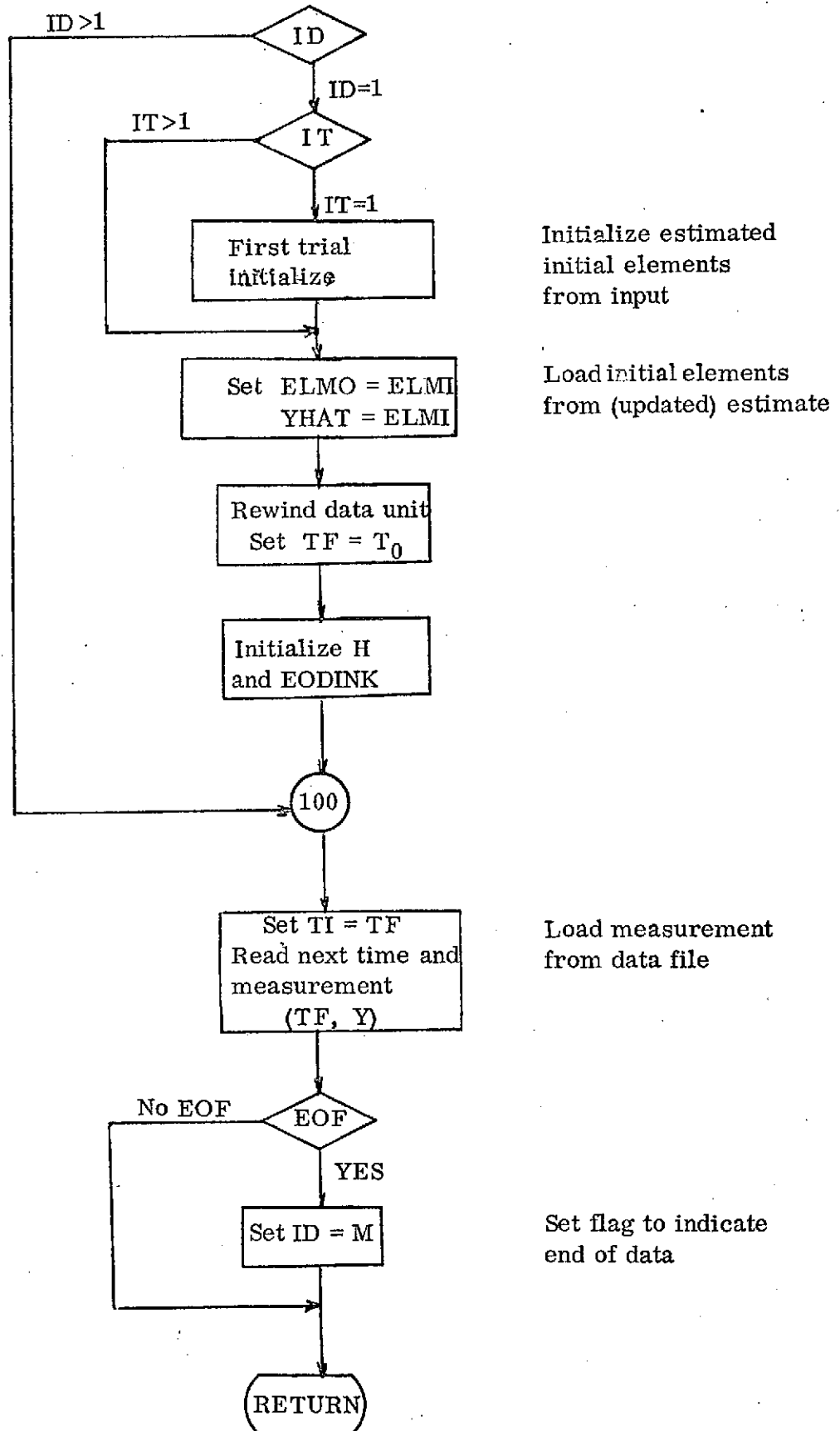
The weighting matrix, Q , used in weighted least squares estimation under the $IQ=3$ option can be converted from Cartesian EE50 variables to method KMETHOD variables by REDATA. It is expected that COVIN will be read in from logical 21 if $IQ=3$, and that COVIN will be the upper (or lower) half of a covariance matrix in the Cartesian state. This is loaded into a filled-out, symmetrical Q and converted to elements by

$$Q_E = \left(\frac{\partial E}{\partial X} \right) Q_X \left(\frac{\partial E}{\partial X} \right)^T$$

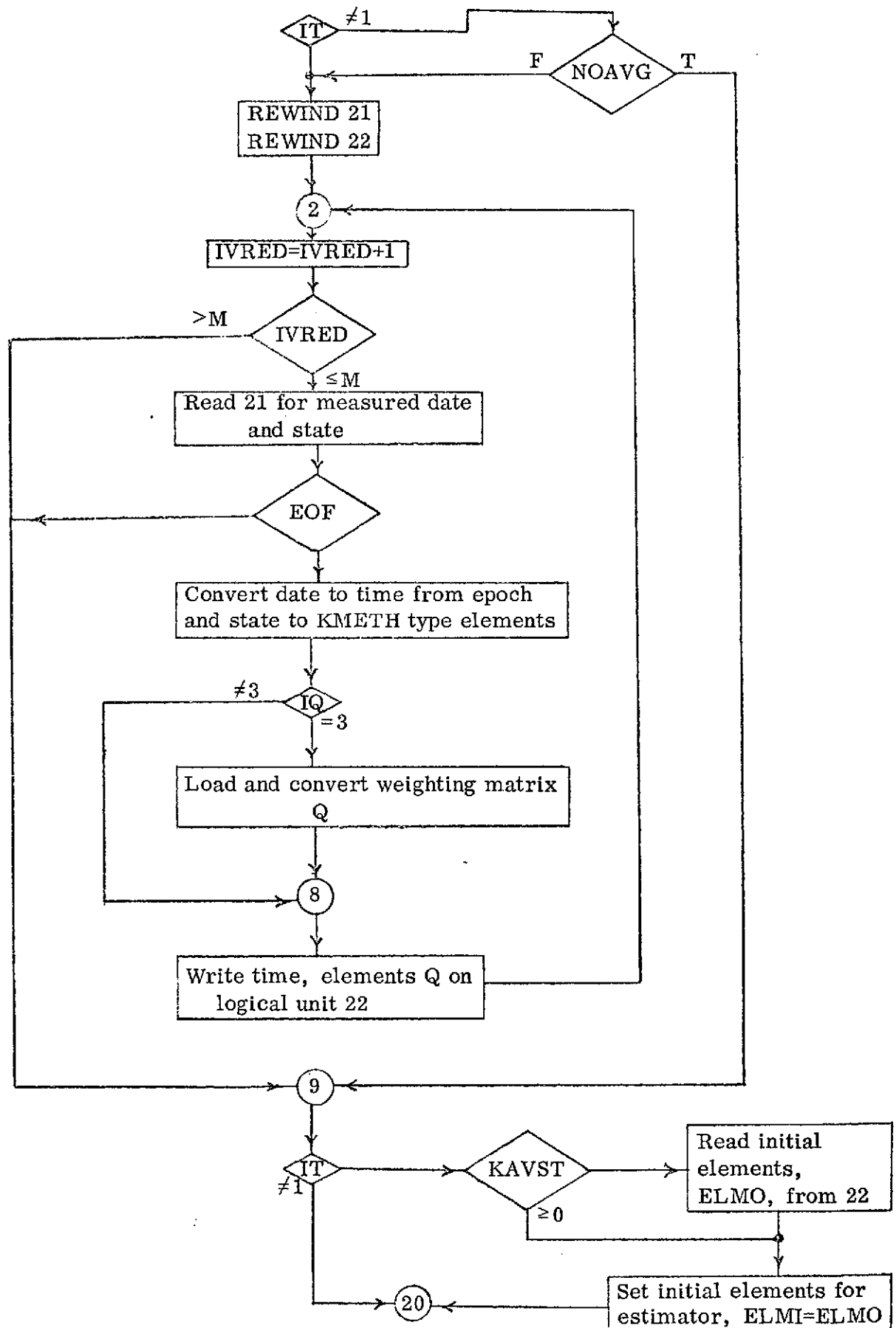
The required partial derivatives are computed by the secant method. Q_E is written onto logical 22 for potential use by subroutines WLS and SQRUT.

The number of measurements written onto 22 will be the smaller of M (input maximum number) and the number found on logical 21.

REDATA Flow Chart (I)



REDATA FLOW CHART (II)
(Initialization Logic)



SUBROUTINE SHIMMY

Calling Sequence: CALL SHIMMY (EM, D, ELMR, R)

Purpose: SHIMMY calculates the rates in the linear variational equations for the variables p , $\text{ecos}(\omega)$, $e \sin(\omega)$, $\omega + f$, i , Ω

Common Blocks Required: CONST, CNTRL, FIELDM, FIELDX, INPUT, INTVAR, INTVRX, IXCUM, PERT.

Subroutines required: MATMPY, ROTATE.

Input / Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	D	3, 3	Calling Operand	Transformation matrix from M50 to position vector coordinates.
I	DFDCX	3, 100	FIELDX (10)	Partials of F w.r.t. harmonic coefficients
I	DVDX 2	3, 3	FIELDX (1)	Gradient matrix
I	ELMR	12	Calling Operand	Orbital elements with sines and cosines
I	EM	6, 3	Calling Operand	Matrix which relates the disturbing acceleration to the orbital element rates
I	GM	12	CONST (5)	Gravitational constants
I	NB	1	IXCOM (101)	Number of coefficients to be studied
O	QATES	600	INTVRX (601)	The rates of the linear variational equations
I	R	1	Calling Operand	Distance to S/C
I	RCART	3	PERT (1)	Disturbing acceleration of S/C in mean of 50

Input / Output (continued)

I	RSW	3	PERT (4)	Disturbing acceleration of S/C in orbital coordinates
I	SELNEQ	3, 3	FIELDM (289)	Transformation matrix from SELNEQ= Q^T to inertial selenographic coordinates.
I	Z	600	INTVRX (1)	Linear variational equation variables

Theory: The equations for the time rates of the orbital elements are represented by:

$$\dot{\mathbf{E}} = \mathbf{F}(\mathbf{E}, \mathbf{C}, \vec{\mathbf{r}}(\mathbf{E}), t),$$

where $\vec{\mathbf{r}}(\mathbf{E})$ represents an implicit dependence upon the position/ velocity vector. These equations may be differentiated to yield equations which represent the effects of small variations in the gravity coefficients upon the subsequent history of the orbital elements;

$$\text{ie.} \quad \frac{\partial}{\partial \mathbf{C}} \left(\dot{\mathbf{E}} \right) = \frac{d}{dt} \frac{\partial \mathbf{E}}{\partial \mathbf{C}} = \left\{ \left(\frac{\partial \mathbf{F}}{\partial \mathbf{E}} \right) + \left(\frac{\partial \mathbf{F}}{\partial \vec{\mathbf{r}}} \right) \frac{\partial \vec{\mathbf{r}}}{\partial \mathbf{E}} \right\} \frac{\partial \mathbf{E}}{\partial \mathbf{C}} + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{C}} \right),$$

where the normal brackets () represent explicit differentiation. The equations of motion can be written in matrix form as follows:

$$\begin{bmatrix} \dot{\mathbf{E}} \\ 6 \times 1 \end{bmatrix} = \begin{bmatrix} \mathbf{N} \\ 6 \times 1 \end{bmatrix} + \begin{bmatrix} \mathbf{M}(\mathbf{E}) \\ 6 \times 3 \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{S} \\ \omega \\ 3 \times 1 \end{bmatrix} = [\mathbf{N}] + [\mathbf{M}(\mathbf{E})] \begin{bmatrix} \mathbf{D}(\mathbf{E}) \\ 3 \times 3 \end{bmatrix} \begin{bmatrix} \mathbf{Q}(t) \\ 3 \times 3 \end{bmatrix} \begin{bmatrix} \vec{\nabla} \mathbf{V} \\ 3 \times 1 \end{bmatrix} = \mathbf{F}$$

where $\mathbf{D}(\mathbf{E})$ is the transformation matrix from the inertial frame to the radial, circumferential, orbit normal orbital coordinate system, $\mathbf{Q}(t)$ is the rotation from selenographic to inertial coordinates, and $\vec{\nabla} \mathbf{V}$ is the gradient of the selenocentric potential function. The vector \mathbf{N} contains the unperturbed (Keplerian) part of the rate of the true anomaly. The 6x6 matrix in curly brackets is evaluated in two parts made up of the explicit partial derivative matrix,

$$\frac{\partial (\dot{p}, \dot{h}, \dot{k}, \dot{u}, \dot{i}, \dot{\Omega})}{\partial (p, h, k, u, i, \Omega)} \quad \text{and the product} \quad \left(\frac{\partial F}{\partial \vec{r}} \right) \left(\frac{\partial \vec{r}}{\partial E} \right), \quad \text{where}$$

$$\left(\frac{\partial F}{\partial \vec{r}} \right) = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} Q \end{bmatrix} \begin{bmatrix} J_B \end{bmatrix} \begin{bmatrix} Q^T \end{bmatrix} \quad \text{and}$$

$$\frac{\partial \vec{r}}{\partial E} = \frac{\partial}{\partial E} \begin{bmatrix} D^T \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \end{bmatrix} = \left(\frac{\partial}{\partial E} \begin{bmatrix} D^T \end{bmatrix} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} D^T \end{bmatrix} \frac{\partial}{\partial E} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \right)$$

The last three columns of $\frac{\partial F}{\partial \vec{r}}$ contain zeros because it is assumed that no velocity-dependent forces are acting. Therefore, the last three rows of $\frac{\partial \vec{r}}{\partial E}$ are not required and set to zero. The 3 X 3 matrix J_B is the Jacobian matrix of the disturbing potential with respect to the body-fixed cartesian coordinates. Finally, the matrix $\left(\frac{\partial F}{\partial C} \right)$ gives the explicit partials of the planetary equations with respect to the gravity coefficients.

Description: The equations discussed above are of the form $Z = AZ + B$.

SHIMMY calculates the matrices A and B, and then uses the input value of Z to calculate the new value of (QATES). The matrices Z and QATES are dimensioned 6 X 100, and represent the quantities $\partial E / \partial C$ and $\partial \dot{E} / \partial C$ respectively, where each column of the matrices corresponds to a different harmonic coefficient of the field. The input quantity NB determines how many coefficients are to be studied, with a maximum of one hundred.

The matrix B represents the quantity $\left(\frac{\partial F}{\partial C} \right)$, discussed above, which may be written as

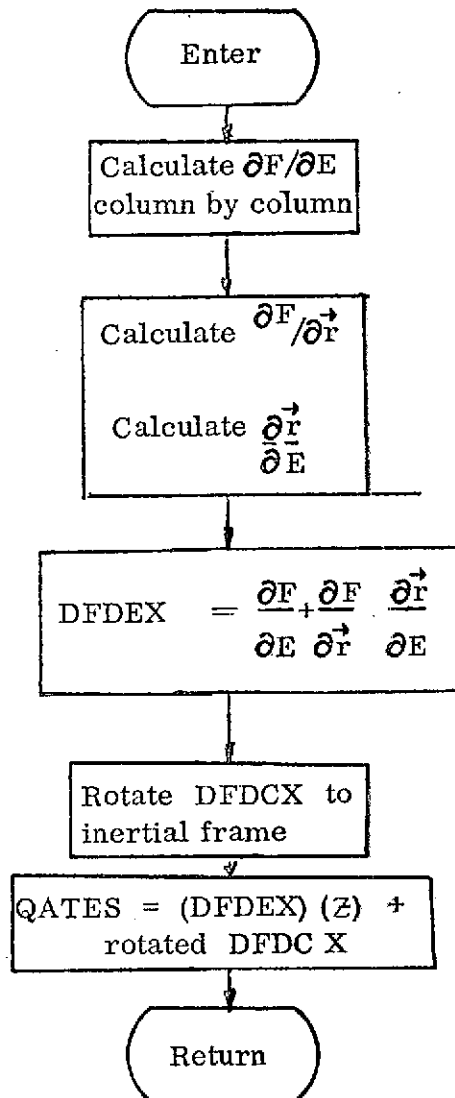
$$\left(\frac{\partial F}{\partial C} \right) = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} Q \end{bmatrix} \begin{bmatrix} \frac{\partial \nabla V}{\partial C} \end{bmatrix},$$

where M is a 6 X 3 matrix which represents the relationship between the perturbing acceleration and the rates of the orbital elements. The matrix is calculated in EQNS, and stored in EM. The transformation matrices $\begin{bmatrix} D \end{bmatrix}$ & $\begin{bmatrix} Q \end{bmatrix}$ are discussed in the theory section and are stored in the 3 x 3 matrices D and SELNEQ respectively. Both of these arrays are calculated outside of SHIMMY. The matrix $\frac{\partial \nabla V}{\partial C}$ is calculated in FIELD2 and stored in DFDCX. For further discussion

of this matrix and the order in which the elements are stored, please see subroutine FIELD2.

The matrix A represents the quantity $\partial F / \partial E$, and may be expressed as $\left(\frac{\partial F}{\partial E} \right) + \left(\frac{\partial F}{\partial \vec{r}} \right) \frac{\partial \vec{r}}{\partial E}$. Both of these quantities represent the product of several matrices, as indicated in the theory section above. In addition to the matrices already used for the calculation of B, the quantities discussed below must be available for the calculation of A. The matrix D2VDX2, calculated in FIELD2, contains the gradient of the force field and is used in the calculation of $(\partial F / \partial \vec{r})$. The vector RSW represents the disturbing acceleration on the S/C in orbital coordinates, and the vector RCART represents the same disturbing acceleration in inertial coordinates. The matrix $\left(\frac{\partial F}{\partial E} \right)$, ie $\left(\frac{\partial \dot{p}, \dot{h}, \dot{k}, \dot{u}, \dot{i}, \dot{\Omega}}{\partial p, h, k, u, i, \Omega} \right)$, is calculated column by column internally through successive matrix multiplications and stored in DFDEX. The matrix and vector multiplications are done in subroutine MATMPY and ROTATE. Once these calculations are completed, the vector Z is inserted into the equation and the vector QATES is output.

SHIMMY FLOW CHART



SUBROUTINE SOLVE

Calling Sequence:

CALL SOLVE

Purpose:

SOLVE solves the equation $W^T W X = Y$ for X , given the vector, Y , and the upper-triangular matrix, W .

Common Blocks Required:

DXCOM, IXCOR

Subroutines Called:

None

Input / Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	QH	NQH, max 105	DXCOM(1)	Square-root information matrix, W
I	HQY	N, max 10	DXCOM(106)	Weighted residual vector, Y
O	DX	N, max 10	DXCOM(116)	Incremental state vector, X
I	N	1	IXCOM(101)	Number of states

Note: The square-root information matrix, QH , is the singly-dimensioned representation of a doubly-dimensioned packed upper-triangular matrix, W .

$$QH = (W_{11}, W_{12}, W_{22}, W_{13}, W_{23}, \dots, W_{nn}, + \text{unused words})$$

Method

The weighted least squares estimation equation can be stated in the form

$$(W^T W) X = Y$$

where W is the upper-triangular square-root information matrix, X is the estimated state increment to be solved for and Y is the weighted residual vector. An upper triangular matrix is a square matrix which has only zeroes below its main diagonal. One obvious solution to the stated problem would be to form $W^T W$, invert it, and multiply Y by it to obtain X . This subroutine uses another method, capitalizing on the fact that W is upper-triangular. We first solve the problem

$$W^T B = Y$$

for the dummy vector, B, by forward substitution. We then solve

$$WX = B$$

by backward substitution to obtain X.

Forward substitution:

$$\underline{W^T B = Y}$$

$$w_{11}b_1 = y_1$$

$$w_{12}b_1 + w_{22}b_2 = y_2$$

$$\vdots$$

$$\sum_{i=1}^j w_{ij}b_i = y_j$$

Solution

$$b_1 = y_1/w_{11}$$

$$b_2 = (y_2 - w_{12}b_1)/w_{22}$$

$$\vdots$$

$$b_j = (y_j - \sum_{i=1}^{j-1} w_{ij}b_i)/w_{jj}$$

Backward Substitution:

$$\underline{WX = B}$$

$$w_{nn}x_n = b_n$$

$$w_{n-1,n-1}x_{n-1} + w_{n-1,n}x_n = b_{n-1}$$

$$\vdots$$

$$\sum_{i=j}^n w_{ji}x_i = b_j$$

Solution

$$x_n = b_n/w_{nn}$$

$$x_{n-1} = (b_{n-1} - w_{n-1,n}x_n)/w_{n-1,n-1}$$

$$\vdots$$

$$x_j = (b_j - \sum_{i=j+1}^n w_{ji}x_i)/w_{jj}$$

Since the zeroes of W do not enter the solution, the potentially non-zero elements of W are stored in a singly-dimensioned array. The coding synthesizes the doubly-dimensioned indices to effect the above procedure as shown.

SUBROUTINE SORDID

Calling Sequence:

CALL SORDID

Purpose:

SORDID sorts the parameter identification array, IX, for FIELD2's required order and prints it out.

Common Blocks Required:

FIELDM, INPUT, IXCOR, STATE

Subroutines Called:

None

Input/Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	ZONL	16	FIELDM(1)	Zonal harmonic coefficients
I	TSRL	16, 17	FIELDM(17)	Longitudinal harmonic coefficients
I	NMOD	1	FIELDM(298)	Maximum modeled N
I	MMOD	1	FIELDM(299)	Maximum modeled M
I	KMETH	1	INPUT(1036)	Trajectory propagation method key
I/O	IX	100	IXCOM(1)	Parameter identification array
I	N	1	IXCOM(101)	Nonzero elements of IX
I	SPC	1	INPUT(198)	Solar pressure coefficient (280)
I	SPC	1	STATE(36)	Solar pressure coefficient (279)

Discussion

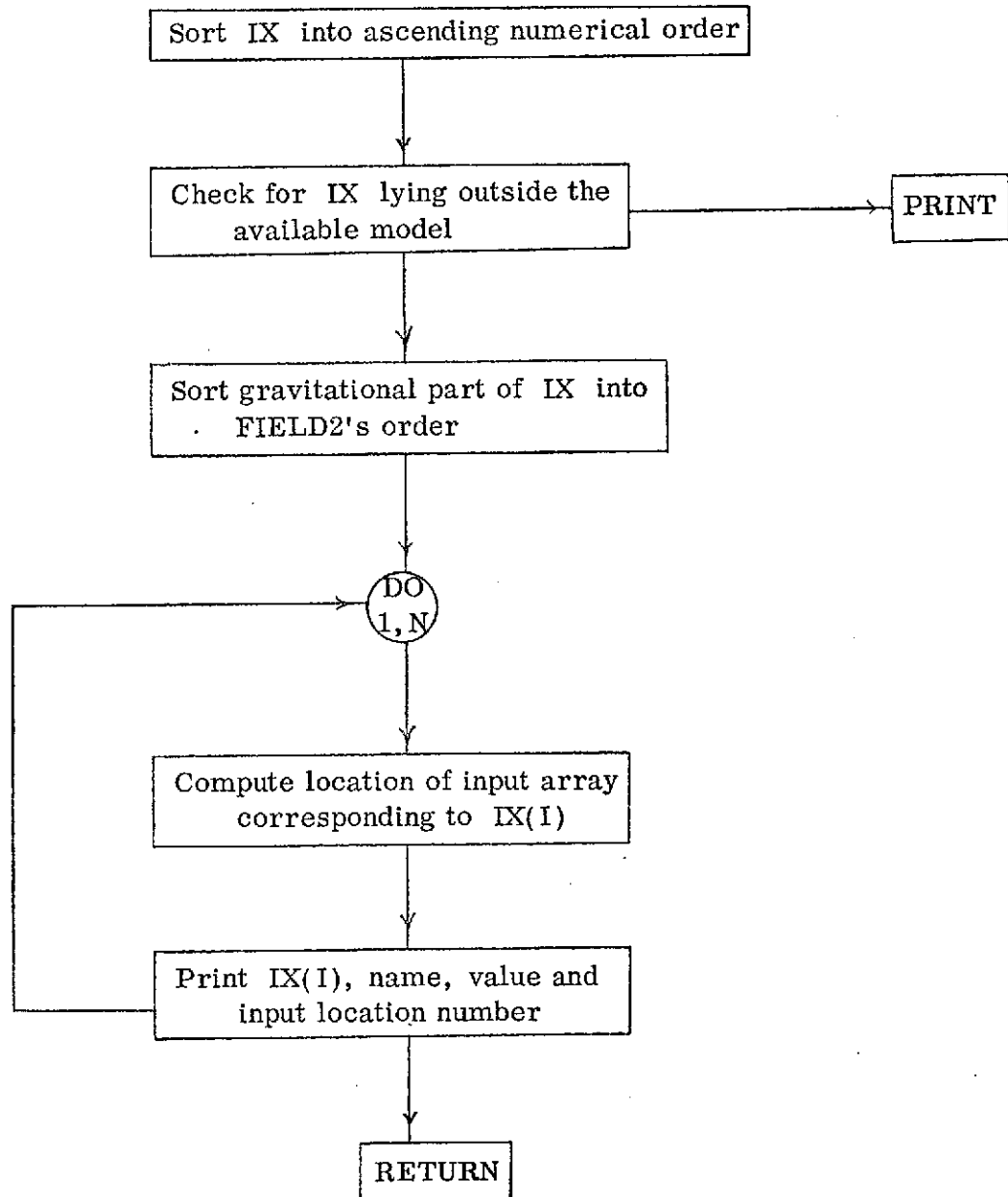
The IX array specifies the parameters to be estimated according to the code:

$$\text{IX(I)} = \begin{cases} 1 \rightarrow 272 & \text{Gravitational parameters} \\ 273 \rightarrow 278 & \text{Initial elements} \\ 279, 280 & \text{Solar pressure coefficients} \end{cases}$$

The code for the gravitational parameters, C_{nm} and S_{nm} , is that IX(I) shall be the element number of the ZONL, TSRL array shown in the following figure. SORDID checks that the input values for IX are valid. That is, IX numbers must lie between 1 and 280 and must not indicate coefficients outside the prescribed model (NMOD, MMOD). Furthermore, subroutine FIELD2 expects a particular order for the selected coefficients in order to efficiently evaluate sensitivities when the option for linear variational equations is selected. That order is $((C(N, M), S(N, M)), M=1, MMOD), N=1, NMOD)$, where coefficients may be omitted, but must not be out of order. SORDID therefore puts the IX array into the order expected by FIELD2 even if the user failed to do so himself. Following the sorting procedure, SORDID prints out the IX number and (for gravitational parameters) the location number in the input array as well as the values residing in the location when SORDID was called. SORDID is called before and after parameter estimation, so parameter values at these times are printed.

0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	
$c_{1,0}$	$c_{1,1}$	$s_{1,1}$	$s_{2,1}$	$s_{3,1}$	$s_{4,1}$	$s_{5,1}$	$s_{6,1}$	$s_{7,1}$	$s_{8,1}$	$s_{9,1}$	$s_{10,1}$	$s_{11,1}$	$s_{12,1}$	$s_{13,1}$	$s_{14,1}$	$s_{15,1}$	1
$c_{2,0}$	$c_{2,1}$	$c_{2,2}$	$s_{2,2}$	$s_{3,2}$	$s_{4,2}$	$s_{5,2}$	$s_{6,2}$	$s_{7,2}$	$s_{8,2}$	$s_{9,2}$	$s_{10,2}$	$s_{11,2}$	$s_{12,2}$	$s_{13,2}$	$s_{14,2}$	$s_{15,2}$	2
$c_{3,0}$	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$	$s_{3,3}$	$s_{4,3}$	$s_{5,3}$	$s_{6,3}$	$s_{7,3}$	$s_{8,3}$	$s_{9,3}$	$s_{10,3}$	$s_{11,3}$	$s_{12,3}$	$s_{13,3}$	$s_{14,3}$	$s_{15,3}$	3
$c_{4,0}$	$c_{4,1}$	$c_{4,2}$	$c_{4,3}$	$c_{4,4}$	$s_{4,4}$	$s_{5,4}$	$s_{6,4}$	$s_{7,4}$	$s_{8,4}$	$s_{9,4}$	$s_{10,4}$	$s_{11,4}$	$s_{12,4}$	$s_{13,4}$	$s_{14,4}$	$s_{15,4}$	4
$c_{5,0}$	$c_{5,1}$	$c_{5,2}$	$c_{5,3}$	$c_{5,4}$	$c_{5,5}$	$s_{5,5}$	$s_{6,5}$	$s_{7,5}$	$s_{8,5}$	$s_{9,5}$	$s_{10,5}$	$s_{11,5}$	$s_{12,5}$	$s_{13,5}$	$s_{14,5}$	$s_{15,5}$	5
$c_{6,0}$	$c_{6,1}$	$c_{6,2}$	$c_{6,3}$	$c_{6,4}$	$c_{6,5}$	$c_{6,6}$	$s_{6,6}$	$s_{7,6}$	$s_{8,6}$	$s_{9,6}$	$s_{10,6}$	$s_{11,6}$	$s_{12,6}$	$s_{13,6}$	$s_{14,6}$	$s_{15,6}$	6
$c_{7,0}$							$c_{7,7}$	$s_{7,7}$									7
$c_{8,0}$								$c_{8,8}$	$s_{8,8}$								8
$c_{9,0}$									$c_{9,9}$	$s_{9,9}$							9
$c_{10,0}$										$c_{10,10}$	$s_{10,10}$						10
$c_{11,0}$											$c_{11,11}$	$s_{11,11}$					11
$c_{12,0}$												$c_{12,12}$	$s_{12,12}$				12
$c_{13,0}$													$c_{13,13}$	$s_{13,13}$			13
$c_{14,0}$														$c_{14,14}$	$s_{14,14}$		14
$c_{15,0}$															$c_{15,15}$	$s_{15,15}$	15

SORDID FLOW CHART



SUBROUTINE SQRUT

Calling Sequence: CALL SQRUT (A, SA, N)

Purpose: SQRUT computes a square root of a positive semi-definite symmetric matrix.

Common Blocks Required: None

Subroutines Called: None

Input / Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	A	(N, N)	Call List	Input matrix, covariance type
O	SA	N*(N+1)/2	Call List	Square root, packed upper-triangular
I	N	1	Call List	Matrix dimension

Method

Given a matrix , A, we seek an upper-triangular matrix, S, such that

$$S^T S = A$$

A recursive algorithm for computing S is

$$s_{ii} = \sqrt{a_{ii} - \sum_{j=1}^{i-1} s_{ij}^2} \quad ; \quad i=1, n$$

$$s_{ji} = \begin{cases} 0 & ; \quad j < i \\ \frac{1}{s_{ii}} \left(a_{ji} - \sum_{k=1}^{i-1} s_{jk} s_{ik} \right) & ; \quad j=i+1, n \end{cases}$$

The output matrix, SA, is "packed" as a singly-dimensioned array.

$$SA = (s_{11}, s_{12}, s_{22}, s_{13}, s_{23}, \dots, s_{nn})$$

SUBROUTINE TUMULT

Calling Sequence: CALL TUMULT (A, B, C, NR, NC, IT)

Purpose: TUMULT multiplies a packed upper-triangular matrix (or its transpose) times a rectangular matrix.

Common Blocks Required: None

Subroutines Called: None

Input / Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I	A	$NR*(NR+1)/2$	Call List	Packed upper-triangular matrix
I	B	$NR * NC$	Call List	Rectangular matrix or vector
O	C	$NR * NC$	Call List	Product matrix
I	NR	1	Call List	Rows of B
I	NC	1	Call List	Columns of B
I	IT	1	Call List	Transpose flag, $IT = \begin{cases} 0 : C = AB \\ 1 : C = A^T B \end{cases}$

Discussion

This subroutine uses the fact that A is upper-triangular to avoid unnecessary multiplications by zeroes in the product AB or $A^T B$. A is packed as a singly-dimensioned array.

$$A = (a_{11}, a_{12}, a_{22}, a_{13}, \dots, a_{nr,nr})$$

SUBROUTINE UPDATX

Calling Sequence:

CALL UPDATX

Purpose:

UPDATX adds the estimated state increment to the appropriate program arrays (estimated state).

Common Blocks Required:

CNSDR, COMIC, DXCOM, FIELDM, IMCOM, INPUT, IXCOS, STATE

Subroutines Called:

None

Input/Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
I/O	ELMI	6	COMIC(1)	Estimated initial elements
I/O	DX	N	DXCOM(116)	Incremental state (upper-loaded)
I/O	X	288	FIELDM(1)	Lunar gravitational field coefficients
I	ICNSDR	20	CNSDR(1)	Considered parameter identifiers
I/O	A	1000	INPUT(1)	Input array
I	IX	N	IXCOM(1)	State component identifiers (upper-loaded)
I	N	1	IXCOM(101)	Number of incremental states
I	IGAIN	1	IMCOM(7)	Reciprocal gain factor
I/O	S(36)	1	STATE(36)	Solar pressure coefficient

Discussion

Each incremental state component in DX is identified by the corresponding element of IX. The program arrays are incremented by DX with a unit gain as follows, according to IX.

<u>If IX(I) is</u>	<u>DX(I) is</u>	<u>DX is added to</u>
$1 \leq IX(I) \leq 271$	a gravitational coefficient	X [IX(I)] in FIELDM
$273 \leq IX(I) \leq 278$	an initial orbital element	ELMI [IX(I)-272] in COMIC
279	a solar pressure coefficient	X(36) in STATE
280	a solar pressure coefficient	A(198) in INPUT

A provision is made for adding the state increment to the estimate with an assigned gain (limit). If IGAIN is input as a number greater than unity,

$$DX = DX/IGAIN$$

Another provision permits exclusion of any incrementing of selected parameters. That is, if any IX(I) is found in the ICNSDR array of "considered" parameters, then the state increment DX(I) corresponding to IX(I) will be set zero. This treatment has the effect of changing the weighting matrix without changing certain parameters during the estimation process.

No flow chart is deemed necessary for this routine.

SUBROUTINE WLS

Calling Sequence:

CALL WLS

Purpose:

WLS is the driver for a weighted least squares estimation process.

Common Blocks Required:

CONST, DXCOM, IMCOM, INPUT, INTVRX, IXCOM, YCOM

Subroutines Called:

MODEL, PHOUSE, PUTINV, REDATA, SOLVE, SQRUT, TUMULT, UPDATX

Input/Output

I/O	Symbolic Name	Program Dimension	Common Block	Definition
O	QH	5650	DXCOM(201)	Square root information matrix
O	HQY	100	DXCOM(101)	Weighted residual vector
O	DX	100	DXCOM(1)	Estimated state increment
I/O	KVAR	1	INPUT(1096)	Sensitivity computation type key
I	KMETH	1	INPUT(1036)	Integration method key
I	NGROPT	1	INPUT(1064)	Gradient treatment key
I	N	1	IXCOM(101)	Number of state components
I	M1	1	IXCOM(102)	Number of observations at each measurement
I	M	1	IXCOM(103)	Number of measurements
O	NQH	1	IXCOM(104)	Number of elements in packed SRIM
I	IQ	1	IXCOM(105)	Weighting type flag 1. no weighting 2. constant Q 3. variable Q
I	ITMAX	1	IXCOM(106)	Maximum number of iterations

I/O	Symbolic Name	Program Dimension	Common Block	Definition
O	ID	1	IXCOM(107)	Measurement number counter
O	IT	1	IXCOM(108)	Iteration counter
I	YHAT	6	YCOM(2)	Estimated measurement counter
I	IGAIN, LTOL	1	IMCOM(7, 8)	Iteration convergence tolerance numbers
I	IM	6	IMCOM(1)	Measurement selector array
I	Y	6	YCOM(9)	Measurement vector
I	Q	36	YCOM(15)	Measurement weighting vector

Discussion

WLS contains the gross logic for a weighted least squares estimation procedure, using a square-root information matrix formulation. The usual form for the WLS estimator is

$$\left(\sum_{i=1}^m H_i^T Q_i^{-1} H_i \right) d\hat{X} = \sum_{i=1}^m H_i^T Q_i^{-1} \tilde{Y}_i$$

where \tilde{Y}_i is the measurement residual vector ($Y - YHAT$) at the i -th measurement, Q_i^{-1} is the measurement weighting matrix, H_i is the measurement sensitivity to changes in the state and $d\hat{X}$ is the estimated state increment to be determined. In the square-root formulation the information matrix (bracketed term) is not formed per se. Instead, a square-root information matrix, W , is accumulated, where W obeys

$$W^T W = \sum_{i=1}^m H_i^T Q_i^{-1} H_i .$$

See the description of W's computation in subroutine PHOUSE. At each measurement, $Q_i^{-1/2} H_i$ is formed. This product is loaded into the QH array, beginning at location NQH+1 for use by PHOUSE in forming W. The weighted residual vector, HQY is accumulated as indicated by the following equation.

$$HQY = \sum_{i=1}^m (Q_i^{-1/2} H_i)^T (Q_i^{-1/2} \tilde{Y}_i)$$

When HQY and W have been accumulated for M measurements, subroutine SOLVE is called to solve

$$(W^T W) d\hat{X} = HQY$$

for $d\hat{X}$ which then is used by UPDATX to increment the estimated state.

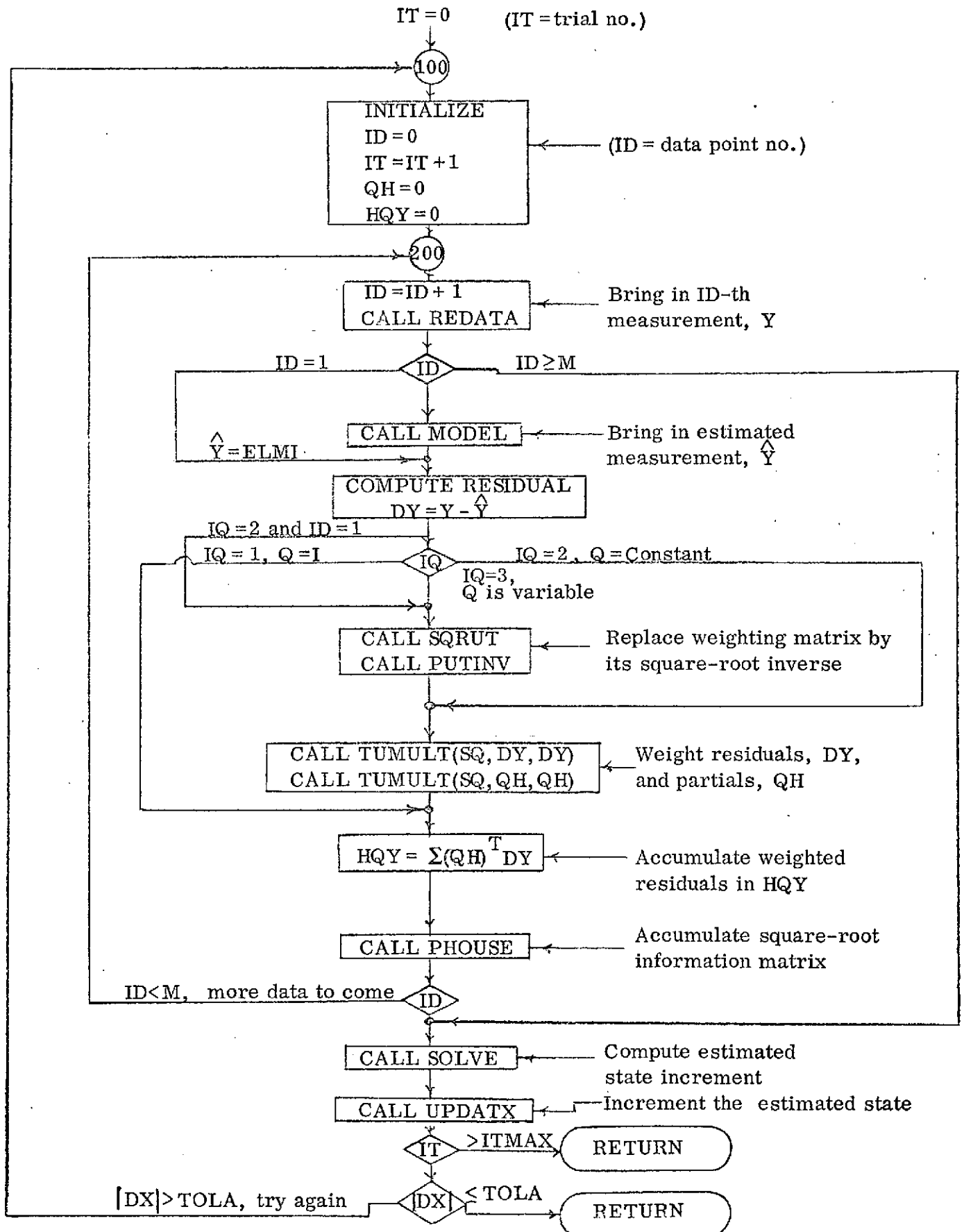
WLS performs ITMAX iterations unless convergence, defined by

$$\sqrt{\sum_{i=1}^N DX_i} \leq \frac{IGAIN}{10^{LTOL}}$$

occurs first. During iterations after the first, the sensitivities (and, therefore, the square-root information matrix) may be held constant at their values on the NGROPT-th iteration. The weighted sensitivities, $Q^{-\frac{1}{2}} H$, are written on logical 23 as they are computed, then read back from 23 as they are needed. Of course, provision is made to avoid redundant computation of sensitivities by setting KVAR and NQ to zero.

When less than 6 elements are used for estimation of state, the residual vector is upper-packed with the necessary M1 components and the weighting matrix is upper-left-packed as specified by the IM array.

WLS Flow Chart



III. THEORETICAL BACKGROUND

3.1 Definition of Symbols

Capital letters will denote arrays (vectors or matrices) unless otherwise stated. Scalars will be denoted by lower case. A subscripted scalar will usually denote a particular element of an array. For example, $W = \{w_{ij}\}$.

$X, \{x_i\}$	State vector (gravitational coefficients, initial orbital elements, solar pressure coefficients)
$Y, \{y_i\}$	Measurement vector (orbital elements)
$H, \{h_{ij}\}$	Sensitivity matrix (of measurement to state)
$Q, \{q_{ij}\}$	Measurement weighting matrix (error covariance matrix)
n	Number of state elements to be solved for
k	Number of observables in one measurement (usually 6)
m	Number of measurements processed
t	Time
$E, \{e_i\}$	Orbital elements (measurement or a function of state and time)
$J, \{j_i\}$	Zonal harmonic coefficients
$C, \{c_{nm}\}$	Tesseral harmonic coefficients in cosine of longitude
$S, \{s_{nm}\}$	Tesseral harmonic coefficients in sine of longitude
d	An operator implying a variation of its operand
\hat{X}	Estimated state
\hat{Y}	Estimated measurement, $Y(\hat{X}, t)$
\tilde{Y}	Measurement residual, $Y - \hat{Y}$
$W, \{w_{ij}\}$	Square root information matrix, upper triangular
()	Superior numbers in parentheses refer to Bibliography

3.2 Weighted Least Squares Parameter Estimation

The fundamental weighted least squares estimation procedure will be derived with no attempt at rigor.

Assume that variations, dX , in the state relate to deviations, dY , in the measurement by the linear relationship (3.1).

$$dY = H dX \quad (3.1)$$

Presuming to have recorded the measurement vector, Y , and to have computed the measurement as a function of the state, $Y(X)$, one may compute the measurement residual, \tilde{Y} .

$$\tilde{Y} = Y - Y(\hat{X}) = Y - \hat{Y} \quad (3.2)$$

If we were to hypothesize a change, $d\hat{X}$, being made in the estimated state, we would expect that, under the linear assumption (3.1), the residual would become

$$\tilde{Y} = Y - (\hat{Y} + H d\hat{X}) \quad (3.3)$$

We seek that $d\hat{X}$ which minimizes the quadratic function, $\tilde{Y}^T Q^{-1} \tilde{Y}$, where Q is a symmetric, positive-definite weighting matrix. One ready derivation of the appropriate $d\hat{X}$ is found by differentiation of the quadratic function with respect to $d\hat{X}$ and equation of the result with zero,

$$-2 \tilde{Y}^T Q^{-1} H = 0 \quad (3.4)$$

followed by solution of the resulting equation for $d\hat{X}$.

$$d\hat{X} = (H^T Q^{-1} H)^{-1} H^T Q^{-1} (Y - \hat{Y}) \quad (3.5)$$

The matrix, $H^T Q^{-1} H$, is called the "information matrix." It must span the state space if it is to possess a unique inverse, although "pseudo-inverses" can always be computed to render solutions to (3.5). Iteration on (3.5) is performed by adding $d\hat{X}$ to \hat{X} and re-computing H and \hat{Y} with the sum before re-applying (3.5). For certain problems, convergence can be obtained without re-computing H during the iteration.

3.3 Square-Root Solution

Certain advantages are to be obtained by using the "square root" form of (3.5). The square root information matrix (SRIM) is defined here as the upper triangular matrix, W , such that (3.6) holds.

$$W^T W = H^T Q^{-1} H \quad (3.6)$$

An upper triangular matrix is a square matrix characterized by having only zeroes below its main diagonal. Conversely, a lower triangular matrix such as W^T has only zeroes above its main diagonal. The factorization of the information matrix can be accomplished by Cholesky decomposition.⁽¹⁾ In special cases, it can also be achieved by forming the square root information matrix rather than the information matrix itself, so that the information matrix need never be factored. The solution of (3.5) is obtained very efficiently, given W , by the method of Banachiewicz.

Beginning with (3.5) in the form of (3.7),

$$(W^T W) d\hat{X} = H^T Q^{-1} \tilde{Y} \equiv D \quad (3.7)$$

we first solve for the dummy vector, B , of (3.8) by forward substitution.

$$W^T B = D \quad (3.8)$$

We then solve (3.9) for $d\hat{X}$ by backward substitution.

$$W d\hat{X} = B \quad (3.9)$$

Denoting an element in the i -th row and j -th column of W by w_{ij} , the forward substitution procedure is (3.10).

$$b_1 = d_1 / w_{11} \quad (3.10.1)$$

$$b_2 = (d_2 - w_{12} b_1) / w_{22} \quad (3.10.2)$$

$$b_j = (d_j - \sum_{i=1}^{j-1} w_{ij} b_i) / w_{jj} \quad (3.10.j)$$

The backward substitution procedure for n state variables is (3.11).

$$dx_n = b_n / w_{nn} \quad (3.11.1)$$

$$dx_{n-1} = (b_{n-1} - w_{n-1,n} dx_n) / w_{n-1,n-1} \quad (3.11.2)$$

$$dx_j = (b_j - \sum_{i=j+1}^n w_{ji} dx_i) / w_{jj} \quad (3.11.j)$$

This forward/backward substitution just described has the nice property that it provides \hat{dX} satisfying (3.7) without explicitly inverting either the information matrix or its square root, thus saving the storage required for the SRIM and its inverse. If both the SRIM and its inverse are required for another reason or if storage is no problem, the substitution method is not quite as advantageous as simple inversion. If it is required, the inverse of the triangular SRIM is very simply obtained by another substitution procedure.⁽²⁾ For sequential information filtering, it is advantageous to retain the SRIM as opposed to replacing it in storage by its inverse.

3.4 Implementing the SRWLS Solution

If successive measurement errors are assumed to be uncorrelated, the right-hand side of (3.7) can be accumulated as a simple vector sum and the SRIM can be accumulated by Householder reduction.^(1,2) If we are trying to estimate n components of state and if we measure km quantities k -at-a-time, the dimensions of the component arrays of (3.7) are as shown in the following table.

Table 3.1 Dimensions

Array	Dimension
\hat{dX}	n
\tilde{Y}	km
Q	$km \times km$
H	$km \times n$
W	$n \times n$

If successive measurements are uncorrelated by the assumed weighting matrix, then

$$Q^{-1} = \begin{bmatrix} Q_1^{-1} & 0 & 0 & \dots & 0 \\ 0 & Q_2^{-1} & 0 & \dots & 0 \\ 0 & 0 & Q_3^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & Q_m^{-1} \end{bmatrix} \quad (3.12)$$

where Q_i^{-1} is a $k \times k$ matrix which weights components of the i -th measurement against each other and against Q_j^{-1} , $j \neq i$. The Q_i^{-1} are themselves positive definite and symmetric, possibly diagonal and possibly equal to each other. We can partition the residual into m k -vectors and H into m ($k \times n$) matrices. If the residual vector for the i -th measurement is \tilde{Y}_i and if H_i is the $k \times n$ sensitivity matrix for the i -th measurement, the right-hand side of (3.7) is the simple vector sum (3.13).

$$H^T Q^{-1} \tilde{Y} = \sum_{i=1}^m H_i^T Q_i^{-1} \tilde{Y}_i \quad (3.13)$$

The information matrix can also be accumulated as a sum.

$$H^T Q^{-1} H = \sum_{i=1}^m H_i^T Q_i^{-1} H_i \quad (3.14)$$

The implemented square root method eliminates about half of the multiplications implied in (3.14) by accumulating the SRIM rather than the information matrix. If W_{i-1} is the upper triangular SRIM after $i-1$ measurements have been included, the SRIM after the i -th measurement must obey (3.15).

$$W_i^T W_i = W_{i-1}^T W_{i-1} + H_i^T Q_i^{-1} H_i \quad (3.15)$$

Equation (3.15) can be written in partitioned form as follows.

$$\begin{bmatrix} W_i^T & \rho \end{bmatrix} \begin{bmatrix} W_i \\ 0 \end{bmatrix} = \begin{bmatrix} W_{i-1}^T & H_i^T Q_i^{-1/2} \end{bmatrix} \begin{bmatrix} W_{i-1} \\ Q_i^{-1/2} H_i \end{bmatrix} \quad (3.16)$$

The computation of W_i consists, then, of finding a means for converting

$$\begin{bmatrix} W_{i-1} \\ Q_i^{-1/2} H_i \end{bmatrix} \Rightarrow \begin{bmatrix} W_i \\ 0 \end{bmatrix} \quad (3.17)$$

where W_{i-1} and W_i are $n \times n$ upper triangular matrices, where $Q_i^{-1/2} H_i$ and 0 are $k \times n$ matrices and where W_i satisfies (3.15). The process which has been implemented is called the Householder reduction algorithm. Since $Q_i^{-1/2}$ and $Q_i^{-1/2} H_i$ must be formed for the HRA, the vector sum in (3.13) can be programmed as in equation (3.18).

$$H^T Q^{-1} \tilde{Y} = \sum_{i=1}^m (Q_i^{-1/2} H_i)^T (Q_i^{-1/2} \tilde{Y}_i) \quad (3.18)$$

The SRIM is initialized to zero to characterize the lack of a priori information.

Application to Lunar Field Determination

The particular application of concern here is, of course, in determination of the harmonic coefficients of the lunar gravitational field. In this application, the state vector, X , consists of those coefficients we choose to solve for plus, optionally, the initial orbital elements (or Cartesian state) and one or more coefficients of the solar radiation pressure model. The state vector might have the following appearance.

$$\text{(example)} \quad X = \begin{bmatrix} j_2 \\ c_{2,1} \\ s_{2,1} \\ 0 \\ e_2 \\ c_{sp} \end{bmatrix} \quad (3.19)$$

The measurement vector, Y , consists of elements of the lunar orbit.

$$Y = \begin{bmatrix} p \\ e \sin \omega \\ e \cos \omega \\ M + \omega \\ i \\ \Omega \end{bmatrix} \quad (3.20)$$

These elements are to be mean elements of the lunar orbit at specified times, referred to the Earth's mean equator and equinox of 1950.0 and having been obtained from short-arc determinations using range and Doppler data. Sensitivities of Y to X are obtained by the Secant Method or, optionally, by integration of variational equations. Propagation of the estimated elements will be carried out by:

1. choosing an initial estimate, \hat{X} , of the state
2. choosing initial elements, probably as the first measurement which would not then be processed as a measurement,
3. integrating the averaged equations of motion across the time span of data.

BIBLIOGRAPHY

1. Kaminski, P. G., "Square Root Filtering and Smoothing for Discrete Processes," SUDAAR No. 427, Guidance and Control Laboratory, Stanford University, Stanford, Calif., July 1971.
2. Schmidt, S. F., "Final Report for Mission Analysis Guidance Study," Report AMA-68-23, Contract NAS5-10566, Mountain View, Calif., January 1969.
3. Schmidt, S. F., "The GSFC-360-Mark-II Program," Report AMA-70-22, Final report for Contract NAS5-20127, Mountain View, Calif., August 1970.
4. Jazwinski, A. H. and Hipkins, C., "Adaptive Estimation for Model Enhancement in Satellite Orbit Determination," Report BTS-TR-72-1, Final Report for Contract NAS5-22011, Bowie, Maryland, July 1972.
5. Conrad, B., "State Estimation with Small Non-Linearities," SUDAAR No. 421, Guidance and Control Laboratory, Stanford University, Stanford, Calif., March 1971.

REFERENCES

1. Lutzky, D. and Schafer, J.; "User's Guide to the Mission Analysis Evaluation and Space Trajectory Operations Program (MAESTRO)," Analytical Mechanics Associates, Inc. Report No. 73-9, March 1973.
2. Uphoff, C. W., Lutzky, D. and Bjorkman, W.S.; "Programmer's Manual for the Mission Analysis Evaluation and Space Trajectory Operations Program (MAESTRO)," Analytical Mechanics Associates, Inc. Report No. 73-10, March 1973.
3. Lutzky, D., Bjorkman, W.S. and Uphoff, C. W.; "Final Report for Radio Astronomy Explorer-B In-Flight Mission Control System Development Effort," Analytical Mechanics Associates, Inc. Report No. 73-8, March 1973.

Appendix A
Derivation of Linear Variational Equations for
the
Method 7 Elements

This appendix contains the expressions for the partial derivative matrices needed in the calculation of the rates for the linear variational equations for the method 7 elements. The method by which these equations are averaged is given at the end of the development.

The equations of motion (the planetary equations in Gauss's form) are

$$\begin{aligned}\dot{\mathbf{E}} &= \mathbf{F}(\mathbf{E}, \mathbf{C}, t) = \mathbf{F}(\mathbf{E}, \mathbf{r}(\mathbf{E}), \mathbf{C}, t) \\ \dot{\mathbf{E}} &= \mathbf{N}(\mathbf{E}) + \mathbf{M}(\mathbf{E}) \begin{bmatrix} \mathbf{R}' \\ \mathbf{S}' \\ \mathbf{W}' \end{bmatrix}\end{aligned}\quad (1)$$

or

$$\dot{\mathbf{E}} = \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{h}} \\ \dot{\mathbf{k}} \\ \dot{\mathbf{u}} \\ \frac{d\mathbf{i}}{dt} \\ \dot{\mathbf{\Omega}} \end{bmatrix}_{6 \times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{\mu p}}{r^2} \\ 0 \\ 0 \end{bmatrix}_{6 \times 1} + \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}_{6 \times 3} \mathbf{M}(\mathbf{E}) \begin{bmatrix} \mathbf{R}' \\ \mathbf{S}' \\ \mathbf{W}' \end{bmatrix}_{3 \times 1}$$

$$q = \sqrt{\frac{\mu}{p}}$$

$$[\mathbf{M}] = \begin{bmatrix} 0 & 2qp \frac{1}{(1+k \cos u + h \sin u)} & 0 \\ q(-\cos u) & q \left[\left(1 + \frac{r}{p}\right) \sin u + \frac{r}{p} h \right] & -qk \frac{r}{p} \sin u \cot i \\ q \sin u & q \left[\left(1 + \frac{r}{p}\right) \cos u + \frac{r}{p} k \right] & q \frac{r}{p} h \sin u \cot i \\ 0 & 0 & -q \frac{r}{p} \sin u \cot i \\ 0 & 0 & q \frac{r}{p} \cos u \\ 0 & 0 & q \frac{r}{p} \frac{\sin u}{\sin i} \end{bmatrix}$$

We wish to describe the effects of small variations in the parameters C on the subsequent motion of the elements. Taking the variation of (1) with respect to the C 's gives

$$\frac{\partial}{\partial C} \dot{E} = \frac{d}{dt} \frac{\partial E}{\partial C} \approx \left\{ \left(\frac{\partial F}{\partial E} \right) + \frac{\partial F}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial E} \right\} \frac{\partial E}{\partial C} + \left(\frac{\partial F}{\partial C} \right) \quad (2)$$

where the normal brackets $()$ represent explicit differentiation and the 6-vector \vec{r} represents the cartesian position and velocity. The first column of the 6×6 matrix $\left(\frac{\partial F}{\partial E} \right)$ is given by

$$\left(\frac{\partial F}{\partial p} \right) = \frac{\partial M}{\partial p} \begin{bmatrix} R \\ S \\ W \end{bmatrix} + \frac{\partial N}{\partial p},$$

and similarly for $\frac{\partial F}{\partial h}$ and $\frac{\partial F}{\partial k}$. The expressions for these first three columns of the matrix $\left(\frac{\partial F}{\partial E} \right)$ are given in tables A1, A2, and A3.

The disturbing acceleration vector, $R' S' W'$, is dependent upon the angular elements u , i , and Ω as well as the time, viz:

$$\begin{bmatrix} R' \\ S' \\ W' \end{bmatrix} = \begin{bmatrix} D \\ 3 \times 3 \end{bmatrix} \begin{bmatrix} Q \\ 3 \times 3 \end{bmatrix} \begin{bmatrix} \nabla V \\ 3 \times 1 \end{bmatrix}$$

where D is the transformation matrix from inertial coordinates to radial, circumferential, orbit normal coordinates. The matrix D and its derivatives with respect to u , i , and Ω are given in table A7. The matrix Q represents the transformation from central planet body-fixed coordinates to the inertial integration frame. ∇V represents the disturbing acceleration due to the central planet non-sphericity and V is the familiar spherical harmonic representation of the disturbing potential:

$$V = \sum_{n=1}^{\infty} \sum_{m=0}^n \frac{R^n}{r^{n+1}} P_n^m(\sin \beta) (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \quad (3)$$

where r , β , and λ are the body-fixed radius, latitude, and longitude of the spacecraft, R is the central planet's mean equatorial radius, and P_n^m are the associated Legendre polynomials. The gradient indicates differentiation with

respect to central-planet-fixed coordinates.

Because the matrix D is a function of i , Ω , and u , the explicit derivatives in equation (2) must contain terms involving $\frac{\partial D}{\partial u}$, $\frac{\partial D}{\partial i}$, and $\frac{\partial D}{\partial \Omega}$. These expressions are given in tables A4, A5, and A6 respectively.

The disturbing acceleration itself, $\vec{\nabla V}$, is treated as a function of the cartesian position vector and is differentiated in the usual way, viz:

$$\frac{\partial \vec{\nabla V}}{\partial \vec{r}} = \left[\frac{\partial \vec{\nabla V}}{\partial \vec{r}'} \right] \left[\frac{\partial \vec{r}'}{\partial \vec{r}} \right] = \left[\frac{\partial \vec{\nabla V}}{\partial \vec{r}'} \right] \left[Q \right]^T$$

where \vec{r}' represents the cartesian position vector in a frame fixed in the central planet. \vec{r}' is just the cartesian vector equivalent of the central body-fixed radius, r , latitude, β , and longitude λ of the disturbing function given in equation (3). It is assumed that there are no velocity-dependent forces and the second term of (2) is

$$\left(\frac{\partial F}{\partial \vec{r}} \right) = \left[M \right] \left[D \right] \left\{ Q J_{\beta} Q^T + J_{3rd} + J_{SP} \right\}$$

where J_{3rd} is the Jacobian of the third body disturbing potential and J_{SP} is the gradient of the solar pressure disturbing force. Other disturbing forces should be treated in the same way.

The remaining term in the curly brackets is the matrix representing the transformation from orbital elements to the cartesian position vector. Since

$$\vec{r} = \left[D \right]^T \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}, \text{ and } r = \frac{p}{1 + h \sin u + k \cos u},$$

the first column of $\frac{\partial \vec{r}}{\partial E}$ is

$$\frac{\partial \vec{r}}{\partial p} = \frac{\partial D^T}{\partial p} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} + D^T \begin{bmatrix} \frac{\partial r}{\partial p} \\ 0 \\ 0 \end{bmatrix},$$

and similarly for the other elements. The total matrix is given by

$$\frac{\partial \vec{r}}{\partial E} = \begin{bmatrix} D^T \begin{bmatrix} r/p \\ 0 \\ 0 \end{bmatrix} & D^T \begin{bmatrix} -\frac{r^2}{p} \sin u \\ 0 \\ 0 \end{bmatrix} & D^T \begin{bmatrix} -\frac{r^2}{p} \cos u \\ 0 \\ 0 \end{bmatrix} & X & \frac{\partial D}{\partial i} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} & \frac{\partial D^T}{\partial} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $X = \frac{\partial D^T}{\partial u} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} + D^T \begin{bmatrix} -\frac{r^2}{p} (h \cos u - k \sin u) \\ 0 \\ 0 \end{bmatrix}$.

The remaining term in equation (2) is the explicit derivative of the "plant" with respect to the parameters to be estimated. This differentiation is accomplished for the spherical harmonic coefficients by evaluation of the disturbing function with the desired coefficient set to unity and the other coefficients set to zero. This is possible because the harmonic coefficients enter the equations linearly. For efficiency purposes, this procedure is carried out internally in the subroutine (FIELD2) that calculates the disturbing acceleration due to the central planet non-sphericity. The mechanics of this operation are explained in the FIELD2 subroutine description of Reference 2.

Averaging the Linear Variational Equations

In order to average the variational equations we must change variables in the formulation because the short-periodic parts of the derivatives of the fast variable with respect to h and k are not small. To change from u to Ξ in the formulation of the linear variational equations, we simply calculate the two additional terms of the matrix M and their derivatives.

In the first and second columns of the fourth row of M write

$$M_{41} = \sqrt{\frac{p}{\mu}} \left[\frac{Q}{e} (k \cos u + h \sin u) - 2\sqrt{1-e^2} \cdot \left(\frac{r}{p}\right) \right]$$

and

$$M_{42} = -\sqrt{\frac{p}{\mu}} \left[\frac{Q}{e} (k \sin u - h \cos u) \left(1 + \frac{r}{p}\right) \right]$$

in place of the zeros given earlier when the function F_4 was u instead of Ξ .

The derivative matrices are modified similarly and we add to the tables A1 through A6 the following terms.

$$\left(\frac{\partial M}{\partial p} \right)_{41} = \frac{1}{2\sqrt{\mu p}} \left[\frac{Q}{e} (k \cos u + h \sin u) - 2\sqrt{1-e^2} \left(\frac{r}{p} \right) \right]$$

$$\left(\frac{\partial M}{\partial p} \right)_{42} = -\frac{1}{2\sqrt{\mu p}} \left[\frac{Q}{e} (k \sin u - h \cos u) \left(1 + \frac{r}{p}\right) \right]$$

$$\begin{aligned} \left(\frac{\partial M}{\partial h} \right)_{41} = & \sqrt{\frac{p}{\mu}} \left[\frac{d}{dh} \left(\frac{Q}{e} \right) (k \cos u + h \sin u) + \frac{Q}{e} \sin u + 2 \left(\frac{r}{p} \right) \frac{h}{\sqrt{1-e^2}} \right. \\ & \left. - 2\sqrt{1-e^2} \frac{d}{dh} \left(\frac{r}{p} \right) \right] \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial M}{\partial h} \right)_{42} = & -\sqrt{\frac{p}{\mu}} \left[\frac{d}{dh} \left(\frac{Q}{e} \right) (k \sin u - h \cos u) \left(1 + \frac{r}{p}\right) - \frac{Q}{e} \cos u \left(1 + \frac{r}{p}\right) \right. \\ & \left. + \frac{Q}{e} (k \sin u - h \cos u) \frac{d}{dh} \left(\frac{r}{p} \right) \right] \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial M}{\partial k} \right)_{41} = & \sqrt{\frac{p}{\mu}} \left[\frac{d}{dk} \left(\frac{Q}{e} \right) (k \cos u + h \sin u) + \frac{Q}{e} \cos u \right. \\ & \left. + 2 \left(\frac{r}{p} \right) \frac{k}{\sqrt{1-e^2}} - 2\sqrt{1-e^2} \frac{d}{dk} \left(\frac{r}{p} \right) \right] \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial M}{\partial k} \right)_{42} = & -\sqrt{\frac{p}{\mu}} \left[\frac{d}{dk} \left(\frac{Q}{e} \right) (k \sin u - h \cos u) \left(1 + \frac{r}{p}\right) + \frac{Q}{e} \sin u \left(1 + \frac{r}{p}\right) \right. \\ & \left. + \frac{Q}{e} (k \sin u - h \cos u) \frac{d}{dk} \left(\frac{r}{p} \right) \right] \end{aligned}$$

$$\left(\frac{\partial M}{\partial u} \right)_{41} = \sqrt{\frac{p}{\mu}} \left[\frac{Q}{e} (h \cos u - k \sin u) - 2\sqrt{1-e^2} \frac{d}{du} \left(\frac{r}{p} \right) \right]$$

$$\left(\frac{\partial M}{\partial u} \right)_{42} = -\sqrt{\frac{p}{\mu}} \left[\frac{Q}{e} (k \cos u + h \sin u) \left(1 + \frac{r}{p}\right) + \frac{Q}{e} (k \sin u - h \cos u) \frac{d}{dh} \left(\frac{r}{p} \right) \right]$$

where

$$\frac{Q}{e} = -\frac{1}{\sqrt{1-e^2} + 1}, \quad \frac{d}{dh} \left(\frac{Q}{e} \right) = -\frac{d}{de} \left(\frac{Q}{e} \right) \cdot \frac{h}{\sqrt{1-e^2}}, \quad \frac{d}{dk} \left(\frac{Q}{e} \right) = -\frac{d}{de} \left(\frac{Q}{e} \right) \frac{k}{\sqrt{1-e^2}},$$

and

$$\frac{d}{de} \left(\frac{Q}{e} \right) = -\frac{e}{\sqrt{1-e^2}} \left(\frac{Q}{e} \right)^2.$$

Also

$$\frac{d}{dh} \left(\frac{r}{p} \right) = -\sin u \left(\frac{r}{p} \right)^2$$

$$\frac{d}{dk} \left(\frac{r}{p} \right) = -\cos u \left(\frac{r}{p} \right)^2$$

$$\frac{d}{du} \left(\frac{r}{p} \right) = (k \sin u - h \cos u) \left(\frac{r}{p} \right)^2,$$

and

$$\frac{r}{p} \neq f(p) \text{ so } \frac{d}{dp} \left(\frac{r}{p} \right) = 0.$$

Now, with the above modifications, we have written the linear variational equations for a system whose fast variable is $\Xi = \omega + M$ instead of $u = \omega + f$ but the right hand side is still in terms of u . The solution to these equations will involve $\frac{\partial \Xi}{\partial C}$ rather than $\frac{\partial u}{\partial C}$ and we must convert by writing

$$\frac{\partial u}{\partial C} = \frac{\partial \Xi}{\partial C} \cdot \frac{\partial u}{\partial \Xi} = \frac{\partial \Xi}{\partial C} \cdot \frac{\sqrt{\mu p}}{n r^2}$$

whenever $\frac{\partial u}{\partial C}$ is needed.

The only remaining modification is to change the term $\sqrt{\mu p}/r^2$ to $\sqrt{\mu/a^3}$ in accordance with the equations for the fast parts of the variables u and Ξ respectively.

The fourth equation now reads

$$\dot{\Xi} = \sqrt{\frac{\mu(1-e^2)^3}{p^3}} + \sum_{j=1}^3 M_{4j}^{(RSW)}{}_j \quad (4)$$

and, if we say that $n = \sqrt{\mu/a^3}$,

$$\frac{\partial \dot{\Xi}}{\partial C} = \frac{\partial n}{\partial p} \frac{\partial p}{\partial C} + \frac{\partial n}{\partial h} \frac{\partial h}{\partial C} + \frac{\partial n}{\partial k} \frac{\partial k}{\partial C} + \frac{\partial}{\partial C} \left\{ \sum_{j=1}^3 M_{4j} (RSW)_j \right\}.$$

The first three terms average to

$$\frac{\partial \bar{n}}{\partial C} = \frac{\partial \bar{n}}{\partial \bar{p}} \frac{\partial \bar{p}}{\partial C} + \frac{\partial \bar{n}}{\partial \bar{h}} \frac{\partial \bar{h}}{\partial C} + \frac{\partial \bar{n}}{\partial \bar{k}} \frac{\partial \bar{k}}{\partial C}$$

and this large part of the variation of the fast variable can be added after the averaging integration with the help of

$$\frac{\partial \bar{n}}{\partial \bar{p}} = -\frac{3}{2} \frac{n}{(1-e^2)}, \quad \frac{\partial \bar{n}}{\partial \bar{h}} = -\frac{3hn}{(1-e^2)}, \quad \frac{\partial \bar{n}}{\partial \bar{k}} = -\frac{3kn}{(1-e^2)}.$$

The variational equations then, are averaged in a way that is exactly analogous to the averaging of the method 8 variables. After the numerical averaging of the perturbation terms, the derivative of the average mean motion is added to the rate of the fourth variable and the averaging of the mean motion is taken outside the numerical averaging algorithm.

Table A 1

$$\begin{aligned}
 8 - v \left(\frac{\partial F}{\partial p} \right) = & \begin{bmatrix} 0 & 3 \sqrt{\frac{p}{u}} \left(\frac{r}{p} \right) & 0 \\ \frac{-\cos u}{2 \sqrt{\mu p}} & \frac{\left[\left(1 + \frac{r}{p} \right) \sin u + \frac{r}{p} h \right]}{2 \mu p} & \frac{-k \left(\frac{r}{p} \right) \sin u \cot i}{2 \sqrt{\mu p}} \\ \frac{\sin u}{2 \sqrt{\mu p}} & \frac{\left[\left(1 + \frac{r}{p} \right) \cos u + \frac{r}{p} k \right]}{2 \sqrt{\mu p}} & \frac{\left(\frac{r}{p} \right) h \sin u \cot i}{2 \sqrt{\mu p}} \\ 0 & 0 & \frac{-\left(\frac{r}{p} \right) \sin u \cot i}{2 \sqrt{\mu p}} \\ 0 & 0 & \frac{\left(\frac{r}{p} \right) \cos u}{2 \sqrt{\mu p}} \\ 0 & 0 & \frac{\left(\frac{r}{p} \right) \sin u}{2 \sqrt{\mu p}} \end{bmatrix} \begin{bmatrix} R' \\ S' \\ W' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{3}{2} \sqrt{\frac{\mu}{p}} / r^2 \\ 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

Table A 2

$$\begin{aligned}
 & \frac{\partial F}{\partial h} = \begin{bmatrix} 0 & 2qp(-\sin u)\left(\frac{r}{p}\right)^2 & & & \\ 0 & q\left[(-\sin^2 u)\left(\frac{r}{p}\right)^2 + \frac{r}{p} - \sin u\left(\frac{r}{p}\right)^2 h\right] & + qk\sin^2 u\left(\frac{r}{p}\right)^2 \cot i & & \\ 0 & q\left[-\sin u \cos u \frac{r}{p}^2 + k(-\sin u)\left(\frac{r}{p}\right)^2\right] & q\sin u \cot i\left[\frac{r}{p} + h(-\sin u)\left(\frac{r}{p}\right)^2\right] & & \\ 0 & 0 & -q(-\sin^2 u)\left(\frac{r}{p}\right)^2 \cot i & & \\ 0 & 0 & q(-\sin u \cos u)\left(\frac{r}{p}\right)^2 & & \\ 0 & 0 & \frac{q(-\sin^2 u)\left(\frac{r}{p}\right)^2}{\sin i} & & \end{bmatrix} \begin{bmatrix} R' \\ S' \\ W' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{2\sqrt{\frac{\mu}{p}} \sin u}{r} \end{bmatrix}
 \end{aligned}$$

Note: $q = \sqrt{\frac{p}{\mu}}$

Table A 3

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$$\frac{\partial F}{\partial k} =$$

$$\begin{bmatrix} 0 & 2pq(-\cos u) \left(\frac{r}{p}\right)^2 & 0 \\ 0 & q \left[(-\cos u \sin u) \left(\frac{r}{p}\right)^2 - \cos u \left(\frac{r}{p}\right)^2 h \right] & -q \sin u \cot i \left[\frac{r}{p} - k \cos u \left(\frac{r}{p}\right)^2 \right] \\ 0 & q \left[(-\cos^2 u) \left(\frac{r}{p}\right)^2 + \frac{r}{p} - k \cos u \left(\frac{r}{p}\right)^2 \right] & q h \cot i \sin u (-\cos u \left(\frac{r}{p}\right)^2) \\ 0 & 0 & +q \sin u \cos u \cot i \left(\frac{r}{p}\right)^2 \\ 0 & 0 & -q \cos^2 u \left(\frac{r}{p}\right)^2 \\ 0 & 0 & -q \frac{\sin u \cos u \left(\frac{r}{p}\right)^2}{\sin i} \end{bmatrix}$$

$$\begin{bmatrix} R' \\ S' \\ W' \end{bmatrix}$$

$$+ \frac{\partial N}{\partial k}$$

$$\frac{\partial N}{\partial k} =$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \frac{\cos u}{p} \\ \frac{r}{r} \\ 0 \end{bmatrix}$$

Table A 4

$$\frac{\partial F}{\partial u} = \frac{\partial M}{\partial u} D Q \overrightarrow{\nabla V} + M \frac{\partial D}{\partial u} Q \overrightarrow{\nabla V}$$

$$\frac{\partial M}{\partial u} = \begin{bmatrix} 0 & 2qp (k \sin u - h \cos u) \left(\frac{r}{p}\right)^2 & 0 \\ q \sin u & q \left[(k \sin u - h \cos u) \left(\frac{r}{p}\right)^2 \sin u + \left(1 + \frac{r}{p}\right) \cos u + h(k \sin u - h \cos u) \left(\frac{r}{p}\right)^2 \right] & -qk \cot i \frac{\partial}{\partial u} \left(\frac{r}{p} \sin u \right) \\ q \cos u & q \left[(k \sin u - h \cos u) \left(\frac{r}{p}\right)^2 \cos u - \left(1 + \frac{r}{p}\right) + k(k \sin u - h \cos u) \left(\frac{r}{p}\right)^2 \right] & qh \cot i \frac{\partial}{\partial u} \left(\frac{r}{p} \sin u \right) \\ 0 & 0 & -q \cot i \frac{\partial}{\partial u} \left(\frac{r}{p} \sin u \right) \\ 0 & 0 & q \frac{\partial}{\partial u} \left(\frac{r}{p} \cos u \right) \\ 0 & 0 & q \frac{1}{\sin i} \frac{\partial}{\partial u} \left(\frac{r}{p} \sin u \right) \end{bmatrix}$$

where $\frac{\partial}{\partial u} \left(\frac{r}{p} \sin u \right) = (k \sin u - h \cos u) \left(\frac{r}{p}\right)^2 \sin u + \frac{r}{p} \cos u$

$$\frac{\partial}{\partial u} \left(\frac{r}{p} \cos u \right) = (k \sin u - h \cos u) \left(\frac{r}{p}\right)^2 \cos u - \frac{r}{p} \sin u$$

Table A 5

$$\frac{\partial F}{\partial i} = \frac{\partial M}{\partial i} D Q \left[\overrightarrow{\nabla \nabla} \right] + M \frac{\partial D}{\partial i} Q \left[\overrightarrow{\nabla \nabla} \right]$$

$$\frac{\partial M}{\partial i} =$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & + \frac{q k \left(\frac{r}{p} \right) \sin u}{\sin^2 i} \\ 0 & 0 & - \frac{q \left(\frac{r}{p} \right) h \sin u}{\sin^2 i} \\ 0 & 0 & + \frac{q \left(\frac{r}{p} \right) \sin u}{\sin^2 i} \\ 0 & 0 & 0 \\ 0 & 0 & - \frac{q \left(\frac{r}{p} \right) \sin u}{\sin^2 i} \end{bmatrix}$$

Note: $\frac{\partial N}{\partial i} = 0$

Table A 6

$$\left(\frac{\partial F}{\partial \Omega} \right) = \left[\frac{\partial M}{\partial \Omega} \right] \begin{bmatrix} R' \\ S' \\ W' \end{bmatrix} + \left[\frac{\partial N}{\partial \Omega} \right] + M \left[\frac{\partial D}{\partial \Omega} \right] Q \left[\overrightarrow{\nabla \nabla} \right]$$

$$\frac{\partial M}{\partial \Omega} = 0, \quad \frac{\partial N}{\partial \Omega} = 0$$

Table A 7

$$D = \begin{bmatrix} cu\ co - ci\ so\ su & cu\ so + ci\ co\ su & su\ si \\ -su\ co - ci\ so\ cu & -su\ so + ci\ co\ cu & cu\ si \\ si\ so & -si\ co & ci \end{bmatrix}$$

$$\frac{\partial D}{\partial i} = \begin{bmatrix} si\ so\ su & -si\ co\ su & su\ ci \\ si\ so\ cu & -si\ co\ cu & cu\ ci \\ ci\ so & -ci\ co & -si \end{bmatrix}$$

$$\frac{\partial D}{\partial \Omega} = \begin{bmatrix} -cu\ so - ci\ co\ su & cu\ co - ci\ so\ su & 0 \\ su\ so - ci\ co\ cu & -su\ co - ci\ so\ cu & 0 \\ si\ co & si\ so & 0 \end{bmatrix}$$

$$\frac{\partial D}{\partial u} = \begin{bmatrix} -su\ co - ci\ so\ cu & -su\ so + ci\ co\ cu & cu\ si \\ -cu\ co + ci\ so\ su & -cu\ so - ci\ co\ su & -su\ si \\ 0 & 0 & 0 \end{bmatrix}$$

si = sin i, ci = cos i, so = sin Ω , co = cos Ω , su = sin u, cu = cos u .

$$\left(\frac{\partial N}{\partial p} \right)_4 = -\frac{3}{2} \sqrt{\frac{\mu}{p}} \frac{1}{r^2}$$

$$\left(\frac{\partial N}{\partial k} \right)_4 = 2 \sqrt{\frac{\mu}{p}} \frac{\cos u}{r}$$

$$\left(\frac{\partial N}{\partial h} \right)_4 = 2 \sqrt{\frac{\mu}{p}} \frac{\sin u}{r}$$

$$\left(\frac{\partial N}{\partial u} \right)_4 = 2 \sqrt{\frac{\mu}{p}} \left(\frac{-k \sin u + h \cos u}{r} \right)$$

all other elements are zero.